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STABILITY OF A SIMPLY SUPPORTED ROD SUBJECTED TO A RANDOM LONGITUDINAL FORCE

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Contract No. NAS8-21111
Control No. DCN 1-7-75 - 20115(IF)
Melpar Job No. 7286

Submitted to

GEORGE C. MARSHALL SPACE FLIGHT CENTER
National Aeronautics and Space Administration
Huntsville, Alabama 35812

FOREWORD

This report was prepared by Melpar, Inc., a subsidiary of Westinghouse Air Brake Company, under Contract NAS8-21111, for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under the technical direction of the Aero-Astrodynamics Laboratory, George C. Marshall Space Flight Center, with Mr. Robert S. Ryan, R-AERO-DD, acting as Project Manager.

This study was executed under the direction of Dr. L. L. Fontenot, Program Manager, with Richard R. Mitchell as Principal Investigator.

ABSTRACT

The stability of a simply supported rod which is subjected to a random longitudinal force is studied in this report. The concepts of moment stability and sample stability are defined and their interrelationships are studied by way of a series of examples. These definitions are then applied to the rod deflection stability problem and sufficient conditions for second moment stability are determined for three types of random longitudinal force: Gaussian white noise, a narrow band noise, and a Gaussian noise with a general power spectral density function satisfying certain conditions. It is then shown that the conditions derived for second moment stability actually yield sample stability. A discussion of sample stability and second moment stability for a first order system and some interesting results concerning the stabilization of an inverted pendulum by subjecting its base to a random vertical displacement are presented in the appendices.

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1. INTRODUCTION

The study of stability of stochastic dynamical systems appears to have originated in the basic work of Andronov, Pontryagin and Witt [1], that appeared in the early 1930's. Since that time, due to the increase in the use of stochastic formulations of general system dynamics, the study of system stability from a stochastic point of view has played an increasingly important role.

The question of stability (instability) in the transverse direction of a simply supported uniform plate which is subjected to a stochastically time varying uniformly distributed end load, has been considered recently by Lomen et al. [2]. In these studies, the authors consider a type of stability which Eringen and Samuels [3] and Samuels [4, 5] refer to as mean square stability. The stability of the plate is defined in terms of the second moment of its deflection and explicit results for the case of Gaussian white noise are presented. Their method of attack is based upon the technique of separation of variables and associated decoupled integral equations, thereby avoiding the pathological properties of Gaussian white noise.

However, for real engineering problems, it is reasonably well accepted that almost-sure sample stability is the goal to achieve, since we desire as many sure (i. e., probability one) facts about the operation of the real system as can be obtained. Unfortunately, sample properties are the most difficult to ascertain in the study of stochastic systems; therefore, the studies devoted to this most deterministic form of stability have lagged somewhat behind relative to the easier to establish stability in the mean. For discrete systems, both Kalman [6] and Bharucha [7] have obtained results on almost-sure stability. Among the first results in the USA for continuous systems can be seen in reference [8]. Further results may be found in references [9] through [15].

Both Bharucha and Kalman noticed a connection between exponential stability of the mean square and almost-sure sample stability for discrete systems. Similar relations are established in reference [9], under slightly different conditions, for continuous systems, and under somewhat more relaxed conditions for diffusion processes in reference [10]. These results are applied in section 4.2 of this report.

Since moments often can be obtained or approximated, it is desirable to determine and use any implications in moment properties that exist for almost-sure stability. The examples presented in section 3 point out that one must seriously stop and consider what one is accepting from instability, or stability, in the mean and mean square as it relates to actual physical systems. The implication is that only almost-sure sample stability can ever be of significance in the study of stochastic models of real systems. If this "deterministic" stability property can be inferred from simply determined mean properties, so much the better. If not, the studies should proceed until sample stability properties are established. Otherwise, the stability of the system can be considered as unknown.

The purpose of this study is to apply the concepts of stochastic stability to a practical engineering problem. To this end, the examples of section 3 play a major role since they tend to unify the subject and smooth the engineer's way through the "terminology barrier" of stochastic stability.

The basic equations which describe the dynamic response of a simply supported rod subjected to a random longitudinal force are derived in section 2. In section 3 the reader is introduced to the idea of stochastic stability; the concepts of moment stability and sample stability are defined and their interrelationships are exhibited by way of a series of examples. The definitions of section 3 are applied in section 4 to the problem of the stability of the simply supported rod presented in section 2. The various analyses of section 4 are classified according to the type of stability they yield and to the type of stochastic longitudinal force used. A further discussion of sample stability and second moment stability for a first order system is included in appendix A and some interesting results concerning the stabilization of an inverted pendulum by subjecting its base to a random vertical displacement are presented in appendix B.

2. FUNDAMENTAL EQUATIONS

2.1 Definition of the Problem

For nearly a decade, the principal applications of random vibration theory have been to vehicles, with particular emphasis on missiles, satellites, and space vehicles. General and theoretical aspects of random vibration, the excitations and responses of flexible space vehicle systems, and the practical problems of data acquisition, establishment of specifications, test equipment, and test procedures are well defined. Analyses that have been conducted, generally include restrictive assumptions such as the neglect of the dynamic stability of the system (the characteristics of which are changing in a random manner). When proceeding in this manner, it is possible to miss completely the real dynamic description of the configuration.

Consider the problem (see figure 2-1) of the transverse vibrations of a straight rod loaded by a randomly time varying uniform longitudinal force. The rod is assumed to be simply supported and of uniform cross section along its length. We will make the usual assumptions in the field of strength of materials; viz., that Hooke's law holds and that plane sections remain plane. We shall investigate the behavior of the transverse deflection of the rod to see if it remains stable in some probabilistic sense. This problem is one of the simplest nontrivial problems of the stability of stochastic systems.

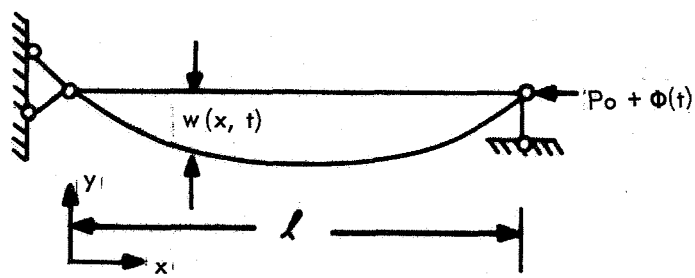

Figure 2-1. Rod Geometry

2.2 Differential Equation of the Problem

We will proceed from the well known equation of the static bending of a rod

$$EI \frac{\partial^2 w}{\partial x^2} + Pw = M \tag{2.1}$$

where $w(x, t)$ is the deflection of the rod, EI is its bending stiffness, and P is the longitudinal force. M is the bending moment resulting from the transverse forces. Since we are interested in the stability problem associated with stochastic end forces and not the forced vibration problem, we will consider the transverse forces to be zero. With this in mind (2.1) becomes, after two differentiations,

$$EI \frac{\partial^4 w}{\partial x^4} + P \frac{\partial^2 w}{\partial x^2} = 0 \tag{2.2}$$

This gives the condition that the sum of the y components of all forces per unit length acting on the rod is in equilibrium.

To arrive at the equation for the transverse vibrations of a rod loaded by the longitudinal force

$$P(t) = P_0 + \Phi(t), \tag{2.3}$$

where P_0 is a constant longitudinal force and $\Phi(t)$ is a mean zero stochastic process, it is necessary to introduce additional terms into (2.2) that take into account the inertial forces.

As in the case of the applied theory of vibrations, we will not include the inertial forces associated with the rotation of the cross sections of the rod with respect to its own principal axes. The influence of longitudinal inertial forces are considered negligible. Note that longitudinal inertial forces can substantially influence the dynamic stability of a rod only in the case where the frequency of the external force is near the longitudinal natural frequencies of the rod (i.e., when the longitudinal vibrations have a resonant character).

With these limiting assumptions, the inertial forces acting on the rod can be reduced to a distributed loading whose magnitude is

$$-m \frac{\partial^2 w}{\partial t^2} \tag{2.4}$$

where m is the mass per unit length of the rod. Thus, we arrive at the differential equation

$$EI \frac{\partial^4 w}{\partial x^4} + (P_0 + \Phi(t)) \frac{\partial^2 w}{\partial x^2} + m \frac{\partial^2 w}{\partial t^2} = 0 \tag{2.5}$$

2.3 Reduction to a System of Ordinary Differential Equations

We shall seek the solution of (2.5) in the form

$$w(x,\ t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi x}{\ell} \tag{2.6}$$

where $f_n(t)$ are functions of time to be determined and ℓ is the length of the rod. One easily sees that the expression (2.6) satisfies the boundary conditions of the problem requiring in the given case that the deflection, together with its second derivative, vanish at the ends of the rod. It should be noted that the "coordinate functions"

$$\sin \frac{n\pi x}{\ell}$$

are of the same form as that of the free vibrations and of the buckling of a simply supported rod.

Substituting expression (2.6) into (2.5) gives

$$\sum_{n=1}^{\infty} \left[m \ddot{f}_n + EI\left(\frac{n\pi}{\ell}\right)^4 f_n - (P_0 + \Phi(t))\left(\frac{n\pi}{\ell}\right)^2 f_n\right] \sin \frac{n\pi x}{\ell} = 0 \tag{2.7}$$

where the dots denote differentiation with respect to time.

For expression (2.6) to satisfy (2.5), it is necessary and sufficient that the quantity in brackets should vanish at any t. In other words, the functions $f_n(t)$ must satisfy the differential equation

$$\ddot{f}_n + \left[\frac{EI}{m}\left(\frac{n\pi}{\ell}\right)^4 - \frac{P_0 + \Phi(t)}{m}\left(\frac{n\pi}{\ell}\right)^2\right] f_n = 0 \tag{2.8}$$

If we define the n^{th} frequency of free vibrations of the unloaded rod and the n^{th} Euler buckling load respectively as

$$\omega_n = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{EI}{m}} \tag{2.9}$$

$$P_n^* = \left(\frac{n\pi}{\ell}\right)^2 EI \tag{2.10}$$

and restrict ourselves here to a consideration of the effect of structural damping where, more precisely, we will consider resistance forces which introduce into (2.8) an additional term containing a first derivative of $f_n(t)$ with respect to time, then (2.8) becomes

$$\ddot{f}_n + 2\beta_n \dot{f}_n + \omega_n^2 \left[1 - \frac{P_0 + \Phi(t)}{P_n^*}\right] f_n = 0 \qquad (2.11)$$

The structural damping coefficients $\beta_n, n = 1, 2, \ldots$ are to be determined experimentally for each case.

For convenience we write equation (2.11) in the form

$$\ddot{f}_n + 2\beta_n \dot{f}_n + \sigma_n^2 \left[1 + \mu_n \Phi(t)\right] f_n = 0 \qquad (2.12)$$

where

$$\sigma_n = \omega_n \sqrt{1 - \frac{P_0}{P_n^*}} \qquad (2.13)$$

$$\mu_n = \frac{1}{P_0 - P_n^*} \qquad (2.14)$$

and σ_n is the frequency of the free vibrations of the rod loaded by a constant longitudinal force P_0.

3. CONCEPTS OF STOCHASTIC STABILITY

3.1 Introduction to Stochastic Stability

The engineer considers the stability of a system to be one of the foremost properties of importance in design. We must, in some way, be assured that a system will operate in a safe mode under a family of conditions that are preselected as representing the system's environment. Ideally, the system is well designed if variations in the environment have no effect on the operational mode of the system. More realistically, any change in the mode of operation caused by changes in the environment should die out as rapidly as possible with the system then returning smoothly to its desired mode of operation. This is, generally, what our intuition tells us when we consider the concept of stability in a practical sense. We then say that the mode of operation of the system is stable.

The engineering procedure for examining the stability of a system is to construct a mathematical model which quite often is a simultaneous set of differential equations -- ordinary or partial, linear or nonlinear, with fixed or time varying coefficients. The dependent variables represent the states of the system and the nature of the environment is reflected in the coefficients and inputs. The stability of the system is then determined in terms of the values (time varying or not) of the coefficients and inputs. Various analytical techniques have been derived to study the system stability when the coefficient parameters are fixed with time, slowly time varying, periodically time varying, and randomly time varying.

When a system design depends upon selecting a specific set of component parameters which are fixed for all time, stability can then be determined by a variety of very powerful classical techniques; Nyquist Diagrams, Root-Locus methods, Routh-Hurwitz criteria, and more recently the extensively studied second method of Liapunov, are all powerful tools that are available to the engineer for the study of system stability. These techniques are all directed towards studying systems that have fixed or time varying parameters when the values of the parameters or the nature of their time variation is known.

In many cases, however, the parameters vary in some fashion unknown to the engineer. As we all know, environments are hardly predictable, and as is becoming more accepted, statistical models of environmental conditions are the only fashion in which we can adequately describe the parameters that characterize the environment. We are led then to a system model consisting of a set of simultaneous differential equations with randomly time varying parameters and inputs. In this report we will be interested in homogeneous systems, i.e., the differential equations will have no right hand side terms. The differential equations used will model a simply supported rod of uniform cross section and various statistical models of the longitudinal force, the environment, will be considered. It should be noted that transverse forces on the rod will not be considered since this leads to right hand side terms.

Basically, a statistical model of an environment is a collection of functions of time where each function describes a possible (or sample) environmental variation. The average or statistical properties of this collection of time functions are governed by joint probability distributions. Usually, only the probability distributions, or perhaps the spectral density and second moments, of a statistical model are specified. In any case, the sample variations are subordinated in most engineering applications. For a given experiment performed on the system, the variation of the environment will lead to a sample variation of the parameters in the differential equation model. Thus a statistical model of the system will consist of a collection of sets of differential equations -- one set for each sample variation of the parameters. Now, the natural question the engineer must ask is what meaning shall he put to the concept of stability as applied to a collection of sets of equations, each with a different set of time varying parameters, when the entire collection of sets of time varying parameters possesses a prescribed set of joint probability distributions? The engineer can look at this problem in two ways -- not unrelated to one another.

In one case, the engineer recognizes that the response of the system, given by the solutions of the differential equations, one solution for each sample variation of the parameters, possesses probability distributions that are generated by the probability distributions of the parameters. Hence, the system response is a random process that possesses average properties.

Thus, given a suitable definition of stability, we may ask that the system be stable in some average sense where the averaging takes place over all sample solutions of the differential equations. For example, we may want the second moments of the system state vector to approach zero asymptotically. This does not tell us what happens in the case of each sample system, but instead, gives us an average response obtained by "watching" all sets of differential equations.

Another possibility, and perhaps the most desirable, is to say that the set of differential equations will be stable in the ordinary deterministic sense for every possible sample variation of the parameters. Thus we would know that no matter what the given environmental variation is, within the given statistical model of the environment, the system is stable. On the surface, this would require that we analyze each sample deterministically by classical techniques. Since there are an infinity of possible environmental variations for any given random model, it is clear that this would be an impossible task to carry out.

Fortunately, as we shall see later, one can make statements about every sample solution of the set of differential equations on the basis of certain average properties of these solutions. This will be taken up in detail in section 4. In order to make precise the ideas that we have described above, we shall define various types of deterministic and stochastic stability and examine their interrelationships by way of examples. In these definitions we will be interested primarily in the linear systems

$$\dot{x} = A(t)x, \quad x(t_0) = x_0, \quad 0 \leq t_0 \leq t \tag{3.1}$$

and

$$\dot{x} = F(t)x, \quad x(t_0) = x_0, \quad 0 \leq t_0 \leq t \tag{3.2}$$

where x is an n-state vector, A(t) is an nxn matrix whose elements are constants and/or known functions of time, and F(t) is an nxn matrix whose elements are constants and/or stochastic processes. It should be noted that definitions 3.1, 3.3, and 3.5 are actually general definitions of stability which also apply to nonlinear systems.

The definitions of moment stability and almost sure sample stability are simple translations of the deterministic definitions of stability. The precise concept of deterministic stability for linear systems is as follows:

Definition 3.1. Stability of a Linear System

The equilibrium state solution $x \equiv 0$ of the system (3.1) is said to be stable if given an arbitrary $\epsilon > 0$, a $\delta > 0$ can be found such that when the initial condition satisfies $||x_0|| < \delta$, it follows that

$$||x(t;x_0, t_0)|| < \epsilon \quad \text{for all } t \geq t_0^{*} \tag{3.3}$$

Simply stated, this definition says that if the system is perturbed less, then it will move less. Thus if we imagine a particle resting in its equilibrium state at the bottom of a spherical bowl and if the particle is perturbed by smaller initial disturbances, then the distance it moves from its rest position will be smaller in magnitude. Notice that if the particle was at rest at the top of a sphere,

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*The notation $x(t;x_0, t_0)$ denotes the solution of (3.1) with initial condition x_0 at time t_0 and $||\cdot||$ denotes a suitable norm.

any disturbance no matter how small will cause large motions of the particle from its initial position. This is the conceptual distinction between a stable and an unstable equilibrium state of a system.*

Definition 3.2. Asymptotic Stability of a Linear System

The equilibrium state solution $x \equiv 0$ of the system (3.1) is said to be asymptotically stable if no matter what the initial condition x_0 is, it follows that

$$\lim_{t \to \infty} ||x(t;x_0, t_0)|| = 0 \tag{3.4}$$

This definition, for linear systems, implies that small disturbances yield small motions, but more than that, (3.4) implies that all disturbances die out. Thus, in the example of a particle in the bottom of a spherical bowl, the particle will eventually return to its initial position. This is the distinction between stability and asymptotic stability. Analytically, we can think of the ordinary undamped oscillation and the damped oscillation as examples possessing respectively a stable equilibrium solution and an asymptotically stable equilibrium solution.

The stochastic analogies to deterministic stability can now be stated quite easily.

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* In this report we will be primarily interested in the stability of a second order differential equation for which the equilibrium state is the origin, i.e., zero position and velocity.

3.2 Moment Stability

Definition 3.3. Stability of the Second Moments for Linear Stochastic Systems

The equilibrium state solution $x \equiv 0$ of the system (3.2) possesses stability of the second moments if given $\epsilon > 0$, there exists a $\delta > 0$ such that for $\| x_0 \| < \delta$, it follows that

$$E\left\{\|x(t; x_0, t_0)\|^2\right\} < \epsilon \quad \text{for all } t \geq t_0 \tag{3.5}$$

We have used the norm $\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ where

$x_1, x_2, \ldots, x_n$ are the components of the vector x.

Definition 3.4 Asymptotic Stability of the Second Moments for Linear Stochastic Systems

The equilibrium state solution $x \equiv 0$ of the linear stochastic system (3.2) possesses asymptotic stability of the second moments if for any x_0,

$$\lim_{t \to \infty} E\left\{\|x(t; x_0, t_0)\|^2\right\} = 0 \tag{3.6}$$

Since moments are relatively easy to determine, these definitions of stochastic stability can be tested analytically in many engineering applications of importance. However, in order to test for moment stability in the laboratory, one would have to operate the system for a large number of runs to obtain a collection of sample responses. Then the average properties of the norms of all the runs would have to be calculated in order to determine what is happening to the second moments. Notice that much information about the characteristics of the individual runs is lost during the averaging procedure so that stability of the second moments will not necessarily tell us whether any given run is stable in the deterministic sense.

Let us consider a few simple examples of first order stochastic systems that will help illustrate some of the definitions above.

Example I.

Consider the simple first order system

$$\dot{x} + a^2 x = 0 \tag{3.7}$$

where a is a Gaussian random variable with zero mean and unit variance. The density function for a is

$$e^{-\frac{a^2}{2}} / \sqrt{2\pi}$$

The solution process for an arbitrary initial condition x_0 at $t_0 = 0$ is

$$\left\{ x_0 e^{-a^2 t}, \ t \in [0, \infty) \right\} \tag{3.8}$$

and the second moment of the solution process is

$$\begin{aligned} E\left\{x^2(t)\right\} &= E\left\{x_0^2 e^{-2a^2 t}\right\} \\ &= \frac{x_0^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2a^2 t - \frac{a^2}{2}} \, da \\ &= \frac{x_0^2}{\sqrt{4t+1}} \longrightarrow 0 \ \text{ as } t \to \infty \ \text{ for any } x_0. \end{aligned} \tag{3.9}$$

Here, we see that this trivial system possesses an equilibrium solution that has asymptotically stable second moments. Indeed we notice that, except for $a = 0$ which occurs with probability zero, all sample solutions approach zero asymptotically as shown in the following graph

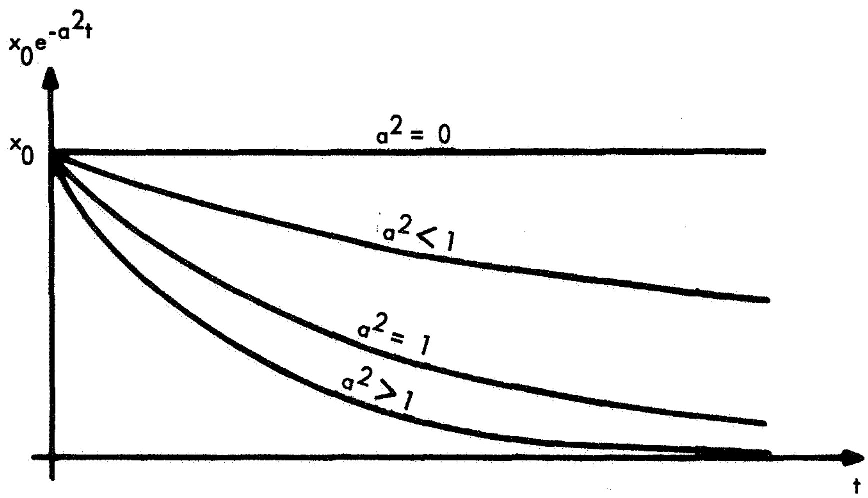

Figure 3-1. Sample Solutions of Equation (3.7)

Example II.

Let us now consider a similar situation for the system

$$\dot{x} + (a+k^2)\,x = 0 \tag{3.10}$$

where k is some fixed but arbitrary constant and a is the random variable of example I.

The solution process is

$$\left\{x_0 e^{-(a+k^2)\,t},\ t \in [0,\infty)\right\} \tag{3.11}$$

The second moments here behave as

$$\begin{aligned} E\left\{x^2(t)\right\} &= x_0^2 e^{-2k^2 t}\, E\left\{e^{-2at}\right\} \\ &= \frac{x_0^2}{\sqrt{2\pi}}\, e^{-2k^2 t} \int_{-\infty}^{\infty} e^{-2at-\frac{a^2}{2}}\, da \\ &= x_0^2\, e^{-2k^2 t + 2t^2} \longrightarrow \infty \text{ as } t \to \infty \text{ for any } x_0 \text{ and any } k \end{aligned} \tag{3.12}$$

Therefore, the system (3.10) possesses unstable second moments. We wish to stress that the moments are unstable for <u>any</u> k. But, suppose we look at the sample solutions

$$x(t) = x_0 e^{-(a+k^2)\,t} \tag{3.13}$$

Obviously, if $a + k^2 > 0$ or $a > -k^2$ the samples will be asymptotically stable as solutions to the differential equation (3.10). But, notice that we can choose k large enough so that $P\left\{a < -k^2\right\}$ is as small as we please. This means that even though the ratio of asymptotically stable solutions to unstable solutions is very large, the moments will still blow up. In fact, one can show examples for which all solutions are stable and yet the moments approach infinity. This is one of the anomalies of using moment properites for the definition of stability. One more situation that can occur is given by the following example.

Example III.

Let us consider the system

$$\dot{x} + [\beta + W(t)]\,x = 0 \tag{3.14}$$

where W(t) is the Gaussian white noise with mean zero and

$$E\{W(t)\ W(t+\tau)\} = S_W \delta(\tau) \tag{3.15}$$

It is shown in appendix A, equation (A-19), that the n^{th} moment $E\{x^n(t)\} \equiv m_n(t)$ of the solution process $\{x(t),\ t\epsilon\ [0, \infty)\}$ satisfies the differential equation

$$\dot{m}_n(t) + [n\beta - \frac{S_W}{2} n\ (n-1)]\ m_n(t) = 0 \tag{3.16}$$

Therefore,

$$m_n(t) = m_n(0)e^{-n[\beta - \frac{S_W}{2}(n-1)]t}$$

Thus for n=2, $\beta > (1/2)S_W$ implies asymptotic stability of the second moment. But for n=4, $\beta > (3/2)S_W$ yields asymptotic stability of the fourth moment. Hence for $(1/2)S_W < \beta < (3/2)S_W$ the system possesses unstable fourth moments and asymptotically stable second moments. Therefore, the engineer can with all justification question what the stability of a moment means to him.

Example IV.

It is not necessary to use the Gaussian white noise to obtain the results of example III. To illustrate this consider the following system which is similar to (3.14)

$$\dot{x} + [\beta + f(t)]x = 0 \tag{3.17}$$

where f(t) is a zero mean stationary ergodic Gaussian stochastic process with a known covariance $\Gamma_f(\tau)$. The solution to (3.17) is

$$x(t) = x_0 e^{-\beta t - \int_0^t f(\tau)\, d\tau} \tag{3.18}$$

and if we let $F(t) = \beta t + \int_0^t f(\tau)\, d\tau$ then F(t) is a Gaussian process with

$$E\{F(t)\} = \beta t$$

$$\sigma_F^2(t) \equiv E\{[F(t) - \beta t]^2\} = \int_0^t \int_0^t \Gamma_f(\tau_1 - \tau_2) d\tau_1 d\tau_2. \tag{3.19}$$

The n^{th} moment $E\{x^n(t)\}$ of the solution process $\{x(t),\ t\epsilon\ [0,\infty)\}$ can be computed directly from (3.18).

$$E\{x^n(t)\} = x_0^n E\{e^{-nF(t)}\}$$
$$= x_0^n \frac{1}{\sqrt{2\pi}\ \sigma_F(t)} \int_{-\infty}^{\infty} e^{-nF}\, e^{\frac{-(F-\beta t)^2}{2\sigma_F^2(t)}}\, dF$$
$$= x_0^n e^{-n\beta t+\sigma_F^2(t)n^2/2} \tag{3.20}$$

Now, for example, consider the covariance

$$\Gamma_f(\tau) = \sigma^2 e^{-|\tau|} \tag{3.21}$$

Substituting (3.21) into (3.19) and integrating gives

$$\sigma_F^2(t) = 2\sigma^2[t+e^{-t}-1] \tag{3.22}$$

which when substituted into (3.20) yields the n^{th} moment

$$E\{x^n(t)\} = x_0^n\ e^{-(\beta-n\sigma^2)\ nt+n^2\sigma^2\ (e^{-t}-1)} \tag{3.23}$$

The necessary and sufficient condition for asymptotic stability of the n^{th} moment is $\beta > n\sigma^2$ so for $2\sigma^2 < \beta < 4\sigma^2$ the system possesses unstable fourth moments and asymptotically stable second moments. Compare this result with that of example III.

We have seen with these simple examples that the moments can take on quite different asymptotic properties and yet, when we have one or the other of the properties holding, we are still not quite sure what we have obtained relative to a real system. Indeed it may happen that a well-behaved system possesses no moments at all ! ! This is illustrated by the next example.

Example V.

Consider the system

$$\dot{x} - a^2 x = 0 \tag{3.24}$$

where a is the Gaussian random variable as defined in example I. The solution process as defined by the system (3.24) is

$$\left\{ x(t) = x_0 e^{a^2 t}, \ t \in [0, \infty) \right\} \tag{3.25}$$

Each sample solution is well behaved although they do increase exponentially. But, the second moments possess a very strange property as we now see.

$$\begin{aligned} E\left\{x^2(t)\right\} &= x_0^2 \, E\left\{e^{2a^2 t}\right\} \\ &= \frac{x_0^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{2a^2 t - \frac{a^2}{2}} \, da \end{aligned} \tag{3.26}$$

For $2t > 1/2$ or $t > 1/4$ the second moment ceases to exist. Indeed for $t > 1/2$ <u>all moments cease to exist.</u>

Once more we see that the properties of the sample solutions are not well reflected by the properties of the moments. Hence, we again question the significance of describing the stability properties of a stochastic system by properties of the moments.

But, we shall not abandon the study of moment properties for, as we shall see later, under certain conditions they do yield significant stability properties.

3.3 Sample Stability

We have indicated that stability in some average sense, say stability of the moments, may not be useful in studying stochastic systems. The question is, what kind of stability property shall we study for stochastic systems that will be useful from the engineer's point of view?

The most desirable property that we could ask for is stability in a deterministic sense. This is not possible, however, for stochastic systems, but almost sure stability or, equivalently, stability with probability one is the closest stochastic analog to deterministic stability. This essentially means that every time we turn the system on, it is stable. The following definitions describe the type of stability that we are referring to.

Definition 3.5. Almost Sure Sample Stability for Linear Stochastic Systems.

The equilibrium state solution $x \equiv 0$ of the linear stochastic system (3.2) is said to be almost surely sample stable if

$$\text{Prob}\,\{\text{given } \epsilon > 0, \text{ there exists } \delta > 0 \text{ such that } \|x_0\| < \delta \text{ implies } \|x(t; x_0, t_0)\| < \epsilon \text{ for all } t \geq t_0\} = 1 \tag{3.27}$$

A definition equivalent to (3.27), which is commonly used and is the analog of definition (3.1), is the following: If, given ϵ, $\epsilon' > 0$, there exists a $\delta > 0$ such that

$$\text{Prob}\,\left\{\sup_{\|x_0\| < \delta}\ \sup_{t \geq t_0} \|x(t; x_0, t_0)\| > \epsilon'\right\} < \epsilon$$

then the equilibrium state solution of (3.2) is almost surely sample stable.

Definition 3.6. Almost Sure Asymptotic Sample Stability for Linear Stochastic Systems.

The equilibrium state solution $x \equiv 0$ of a linear stochastic system is said to be almost surely asymptotically sample stable if

$$\text{Prob}\,\left\{\lim_{t \to \infty} \|x(t; x_0, t_0)\| = 0\right\} = 1 \tag{3.28}$$

A simple example will illustrate the "almost sure" nature of these definitions. Consider the first order differential equation $\dot{x} = ax$ with initial condition $x(0) = x_0$ where the parameter a is a uniformly distributed random variable on the interval

$-1 \le a \le 0$. The sample solutions are decaying exponentials for all values of a except $a = 0$. Since $a = 0$ occurs with probability zero, this system is almost surely asymptotically stable. There are no unstable solutions so we actually have deterministic stability in the sense of definition 3.1 but not deterministic asymptotic stability, definition 3.2.

The equilibrium state solution of the system in example I is almost surely asymptotically sample stable.

It is shown in appendix B that the equilibrium state solutions of examples III and IV are almost surely asymptotically sample stable for

$$\beta > -(1/2)S_W \tag{3.29}$$

and

$$\beta > 0, \tag{3.30}$$

respectively. We recall that for second moment stability of the system of example III, the requirement was $\beta > (1/2)S_W$. Hence we find that for $-(1/2)S_W < \beta < (1/2)S_W$ the second moments will grow to infinity while from (3.29) all samples approach zero. Similarly, for example IV, $0 < \beta < 2\sigma^2$ implies that the second moments grow to infinity while from (3.30) all samples approach zero. Again, the moment properties do not reflect what the system is doing on a sample basis.

Examples II-V illustrate the different stability properties of the moment solutions and the sample solutions. These results are compared in table 3.1 which follows example VI. In particular, they show that the sample solutions may be asymptotically stable while various moments have blown up or are diverging as $t \to \infty$. What about the converse situation? Can we construct an example where the solutions blow up but the moments behave nicely?

Example VI

The problem is to find a differential equation whose solution is unbounded (i.e., no matter how large a number you choose, the value of the solution, at some time, will exceed this number) but the expected value of which remains bounded for all time. In this particular example, which is admittedly somewhat artificial from an engineering point of view, the solution is unbounded as $t \to \infty$ but its expected value converges to x_0, the initial condition.

The idea in constructing this example is to create coefficient samples that look like sequences of rectangular pulses with amplitudes that grow but areas that decrease to zero as $t \to \infty$.

Consider the first order (scalar) differential equation

$$\dot{x} = f_1(t)x, \quad x(0) = x_0 > 0 \tag{3.31}$$

which has the solution

$$x(t) = x_0 e^{\int_0^t f_1(\tau)d\tau} \tag{3.32}$$

Define a sequence of time points c_i which are independent and uniformly distributed random variables over the time intervals I_i where $I_i \equiv (i-1) + \frac{1}{(i+1)^2} \leq c_1 \leq i - \frac{1}{(i+1)^2}$.

Note that I_i is contained in the time interval $[i - 1, i]$.

Let the coefficient $f_1(t)$ consist of a sequence of rectangular pulses with level zero or $\pm a_i$, where $a_i = (i + 1)^2 \log (i + 1)$, and length $(i + 1)^{-2}$ which are positioned about the time points c_i as shown in figure 3-2. See figure 3-3 for integration of $f_1(t)$ with respect to time. Using equation (3-32) we see that this particular sample solution has the form shown in figure 3-4. Notice that the amplitudes of the pulses grow without bound as i increases $(t \to \infty)$ and the area under each pulse is bounded by $\frac{x_0}{i+1} \to 0$ as $i \to \infty$.

To determine an upper bound for the expected value of the $\{x(t), t \in [0, \infty)\}$ process we note that in the time interval $(i - 1, i)$ the $P\{x(t) > x_0\}$ is simply the ratio of the length of the interval over which c_i ranges such that $x(t) > x_0$ to the length of the interval I_i (c_i uniformly distributed).

For $i - 1 \leq t \leq i$

$$P\{x(t) > x_0\} \leq \frac{2}{(i+1)^2 - 2} \tag{3.33}$$

$$P\{x(t) = x_0\} \leq 1 \tag{3.34}$$

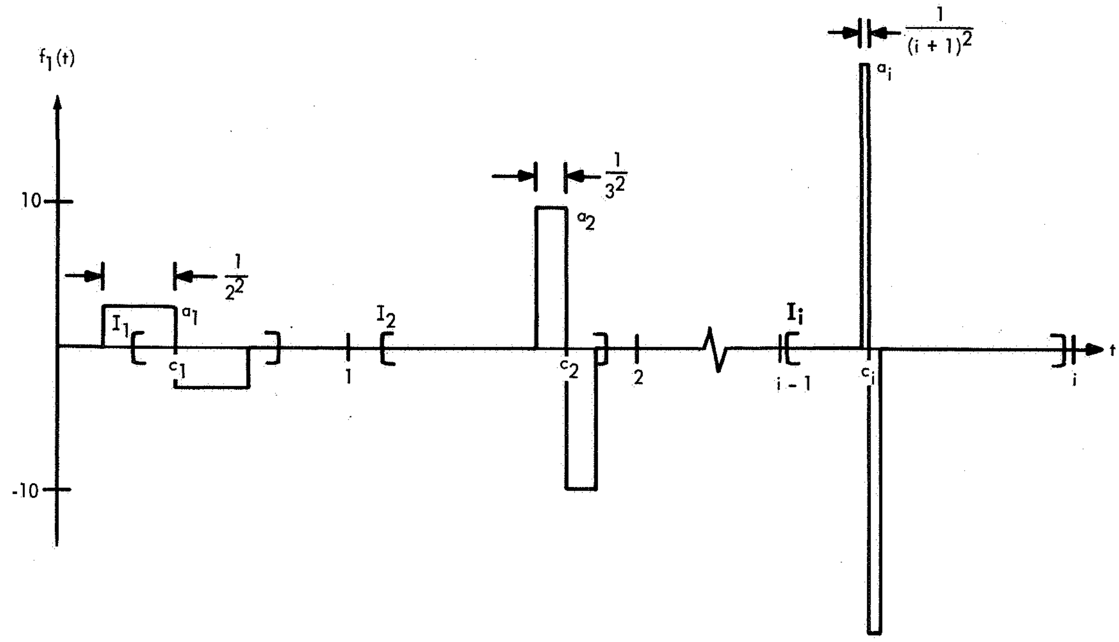

Figure 3-2. Typical Sample of the Coefficient $f_1(t)$.

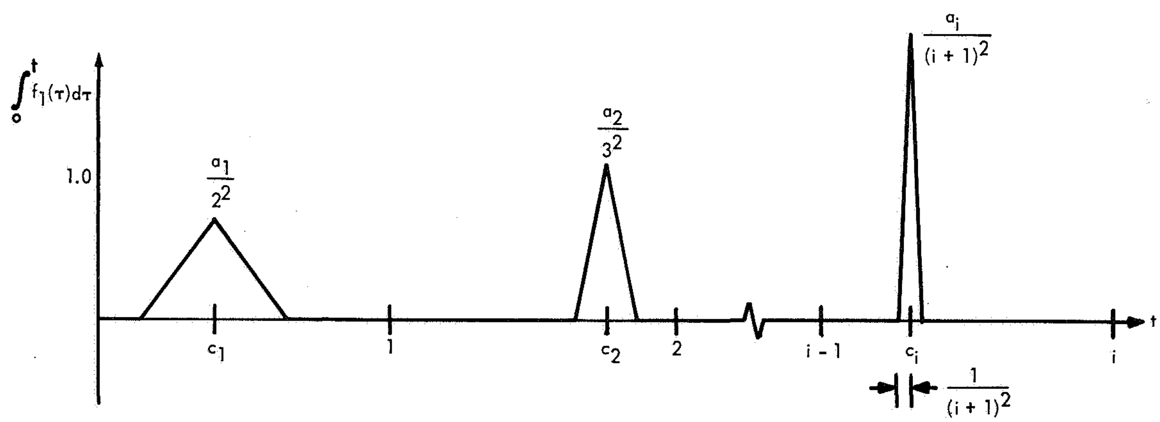

Figure 3-3. Integral of the Sample in Figure 3-2.

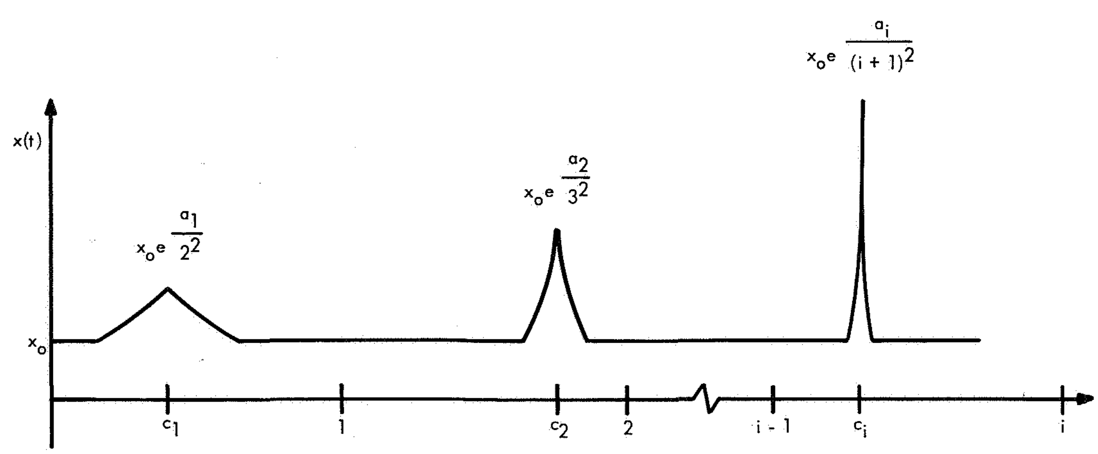

Figure 3-4. Sample Solution of Equation (3.31).

From (3.33) and (3.34) an upper bound for the expected value of the $\{x(t),\ t \in [0, \infty)\}$ process is

$$E\{x(t)\} \le \max_{t \in (i-1,\ i)} [x(t)P\{x(t) > x_0\}] + x_0 P\{x(t) = x_0\}$$

$$\le \frac{2(i+1)x_0}{(i+1)^2 - 2} + x_0,$$

for $t \in (i - 1,\ i)$

$$\rightarrow x_0 \text{ as } i \rightarrow \infty \tag{3.35}$$

$E\{x(t)\}$ can certainly be no less than x_0 so the equality in (3.35) holds, i.e.,

$$\lim_{t \rightarrow \infty} E\{x(t)\} = x_0$$

Thus the samples of the x(t) process are unbounded but $E\{x(t)\}$ is bounded as $t \rightarrow \infty$. In fact, not only is the first moment bounded but it is also stable in the sense that given an $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0| = x_0 < \delta$ implies $E\{x(t)\} < \epsilon$ (let $\delta = 1/3\epsilon$).

TABLE 3.1

COMPARISON OF MOMENT STABILITY AND SAMPLE STABILITY

EXAMPLE	MOMENTS	SAMPLES
I $\quad \dot{x} + a^2 x = 0$	Asymptotically stable	Asymptotically stable
II $\quad \dot{x} + (a + k^2)x = 0$	2nd moment unstable for all k^2	Asymptotically stable for $a + k^2 > 0$

TABLE 3.1 (Continued)

COMPARISON OF MOMENT STABILITY AND SAMPLE STABILITY

EXAMPLE	MOMENTS	SAMPLES
III $\dot{x}+[\beta+W(t)]x=0$	$(1/2)S_W<\beta<(3/2)S_W$ $\begin{cases}\text{2nd moment asymptotically stable}\\ \text{4th moment unstable}\end{cases}$	Asymptotically stable for $\beta>-(1/2)S_W$
IV $\dot{x}+[\beta+f(t)]x=0$	$2\sigma^2<\beta<4\sigma^2$ $\begin{cases}\text{2nd moment asymptotically stable}\\ \text{4th moment unstable}\end{cases}$	Asymptotically stable for $\beta>0$
V $\dot{x}-a^2x=0$	All moments ∞ for $t>1/2$	Increase exponentially
VI $\dot{x}=f_1(t)x$	$E\{x(t)\}$ is stable and $\to x_0$ as $t\to\infty$	Unbounded as $t\to\infty$

Key:

a = Gaussian random variable

k, β, S_W, σ^2 = Constants

$W(t)$ = Gaussian white noise

$f(t)$ = Zero mean stationary ergodic Gaussian stochastic process

$f_1(t)$ = A sequence of steps described in figure 3-2

In this section we have tried to give a capsule idea of what stochastic stability is about, what some of the known results are, what some of the problems are, and how some of the types of stochastic stability are related.

The subject is somewhat new and may not be known to many readers. We suggest that the reader study the examples of this section and appendix A since a great many of the significant features of the subject lie in them. In the sections to follow, we shall use the terminology of this section and discuss specific applications of the ideas presented here.

4. ANALYSES

4.1 Conditions for Second Moment Stability

4.1.1 Asymptotic Stability of the Second Moments

The question of stability in the transverse direction of a simply supported rod of uniform cross section, which is subjected to a stochastically time varying uniform longitudinal force, will be approached in several ways. The approach here is to determine the conditions under which the second moment of the transverse deflection will asymptotically converge to zero. As discussed in the previous section, there is some question as to what this means the sample deflections are doing; however, section 4.2.2 will show that if this convergence is exponential then the samples will also approach zero asymptotically.

Assume the longitudinal force to be a constant plus a zero mean Gaussian white noise. Under this assumption, it is known that the solution of equation (2.12) will be a vector Markov process which enables us to use the many mathematical tools associated with Markov processes. However, all is not rosy. The theory of Markov processes yields a fairly simple method to determine stability in an average sense but, for engineering purposes, this requires the Gaussian white noise assumption which complicates matters for two reasons. First, the mathematics involving the Gaussian white noise is complicated even to the extent of requiring a different calculus when manipulating differential equations with white noise coefficients and, second, it is not true, in general, that similar stability results will be obtained for analyses using white noise and broad spectral band physical noise coefficients. See appendix A for a further discussion of this question.

Since we can obtain the vector Markov property for the solution process of (2.12) by using the Gaussian white noise, let us consider some of the properties of this type of noise. As usually described in the literature, white noise $W(t)$ has a zero mean, a Gaussian distribution, and a power spectral density function with the constant value S_W over all frequencies. It is equivalently characterized as having a covariance of the form

$$E\left\{W(t)\,W(t+\tau)\right\} = S_W \delta(\tau) \tag{4.1}$$

where $\delta(\tau)$ is the Dirac delta function. The Gaussian white noise, in the sense just defined, can be derived from the Brownian motion, or Wiener process $B(t)$. The Brownian motion process with parameter S_W is a Gaussian process whose increments $dB(t) = B(t+dt) - B(t)$ have the mean and variance

$$\begin{aligned} E\left\{dB(t)\right\} &= 0 \\ E\left\{[dB(t)]^2\right\} &= S_W dt. \end{aligned} \tag{4.2}$$

It is known that the samples of B(t) are continuous everywhere but have the unusual property of being nowhere differentiable. However, it can be shown that the Gaussian white noise is representable as the formal derivative of the Brownian motion process,

$$W(t) = \frac{dB(t)}{dt} = \dot{B}(t) \tag{4.3}$$

which does not exist since the samples are not differentiable. To circumvent this apparent paradox, the differential equations will be written as incremental equations and properties such as (4.2) used. The fundamental results here are due to Ito and can be found in reference [16].

Using the vector Markov property of the solution process of (2.12) and following reference [17] we will now derive a stability criterion which is sufficient to ensure asymptotic stability of the second moments of the rod deflection (definition 3.4). If the second moment of the rod deflection is bounded by a decaying exponential then it will converge asymptotically to zero. Since the rod deflection is represented by the infinite series

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{\ell}x \, f_n(t) \tag{4.4}$$

then the second moment is

$$\begin{aligned} E\{w^2(x,t)\} &= \sum_{m,n=1}^{\infty} \sin\frac{m\pi}{\ell}x \, \sin\frac{n\pi}{\ell}x \; E\{f_m(t) f_n(t)\} \\ &= \sum_{m,n=1}^{\infty} \frac{\sin\frac{m\pi}{\ell}x}{\omega_m} \, \frac{\sin\frac{n\pi}{\ell}x}{\omega_n} \; E\{\omega_m f_m(t)\, \omega_n f_n(t)\} \\ &\le \sum_{m,n=1}^{\infty} \frac{\sin\frac{m\pi}{\ell}x}{\omega_m} \, \frac{\sin\frac{n\pi}{\ell}x}{\omega_n} \; [E\{\omega_m^2 f_m^2(t)\} E\{\omega_n^2 f_n^2(t)\}]^{1/2} \end{aligned} \tag{4.5}$$

by the Schwarz inequality. Therefore, if there exist positive constants a and b independent of n such that

$$E\{\omega_n^2 f_n^2(t)\} \le a e^{-bt} \text{ for all } n \text{ and } t \ge 0 \tag{4.6}$$

then

$$E\{w^2(x,t)\} \leq ae^{-bt} \sum_{m,n=1}^{\infty} \frac{\sin\frac{m\pi}{\ell}x}{\omega_m} \frac{\sin\frac{n\pi}{\ell}x}{\omega_n} \tag{4.7}$$

which converges to zero asymptotically since $\sum_{m,n=1}^{\infty} \frac{\sin\frac{m\pi}{\ell}x}{\omega_m} \frac{\sin\frac{n\pi}{\ell}x}{\omega_n}$ converges uniformly and absolutely. Thus, bounding $E\{\omega_n^2 f_n^2(t)\}$ by a decaying exponential for all n will give asymptotic stability of

$$E\{w^2(x,t)\}.$$

The differential equation for $f_n(t)$, (2.12), is

$$\ddot{f}_n(t) + 2\beta_n \dot{f}_n(t) + \sigma_n^2 [1 + \mu_n \Phi(t)] f_n(t) = 0 \tag{4.8}$$

Letting $\mu_n \Phi(t)$ be the Gaussian white noise $\dot{B}(t)$ and putting (4.8) in state vector form by the change of variables

$$\begin{aligned} x_1(t) &= f_n(t) \\ x_2(t) &= \dot{x}_1(t) \end{aligned} \tag{4.9}$$

gives

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\beta_n x_2 - \sigma_n^2 [1+\dot{B}(t)] x_1 \end{aligned} \tag{4.10}$$

Let x be the state vector with components x_1 and x_2. Then for states x, ξ and times $s<t$, the conditional probability density function $p(x,t|\xi,s)$ satisfies the Fokker Planck equation

$$\frac{\partial}{\partial t} p(x,t|\xi,s) = \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_i \partial x_j} [b_{ij}(x,t) p(x,t|\xi,s)]$$

$$- \sum_{i=1}^{2} \frac{\partial}{\partial x_i} [a_i (x,t)\, p(x,t|\xi,s)] \qquad (4.11)$$

where $a_i(x,t)$ and $b_{ij}(x,t)$ are the first and second derivate moments defined by the following conditional expected values

$$a_i(x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{x_i(t+dt) - x_i(t) \,|\, x(t) = x\} \equiv \lim_{dt \to 0} \frac{1}{dt} E\{dx_i(t) \,|\, x(t) = x\} \qquad (4.12)$$

$$b_{ij}(x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{[x_i(t+dt) - x_i(t)]\,[x_j(t+dt) - x_j(t)] \,|\, x(t) = x\}$$

$$\equiv \lim_{dt \to 0} \frac{1}{dt} E\{dx_i(t)\, dx_j(t) \,|\, x(t) = x\} \qquad (4.13)$$

Writing (4.10) as a system of incremental equations

$$dx_1(t) = x_2(t)\, dt \qquad (4.14)$$

$$dx_2(t) = -[2\beta_n x_2(t) + \sigma_n^2 x_1(t)]\, dt - \sigma_n^2 x_1(t)\, dB(t)$$

and substituting this into (4.12) and (4.13) we obtain the first and second derivate moments.

$$a_1 (x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{x_2 (t)\, dt \mid x (t) = x\}$$

$$= \lim_{dt \to 0} \frac{1}{dt} x_2 (t)\, dt$$

$$= x_2 (t) \tag{4.15}$$

$$a_2 (x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{- [2\beta_n x_2 (t) + \sigma_n^2 x_1 (t)]\, dt - \sigma_n^2 x_1 (t)\, dB (t) \mid x (t) = x\}$$

$$= \lim_{dt \to 0} \frac{1}{dt} \left(- [2\beta_n x_2 (t) + \sigma_n^2 x_1 (t)]\, dt - \sigma_n^2 x_1 (t)\, E\{dB (t)\}\right)$$

$$= -2\beta_n x_2 (t) - \sigma_n^2 x_1 (t) \qquad \text{(See equation (4.2).)} \tag{4.16}$$

$$b_{11} (x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{[x_2 (t)\, dt]^2 \mid x (t) = x\}$$

$$= \lim_{dt \to 0} \frac{1}{dt} x_2^2 (t)\, (dt)^2$$

$$= 0 \tag{4.17}$$

$$b_{12}(x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{ [x_2(t)\,dt][-(2\beta_n x_2(t) + \sigma_n^2 x_1(t))\,dt - \sigma_n^2 x_1(t)\,dB(t)] \mid x(t) = x\}$$

$$= \lim_{dt \to 0} \frac{1}{dt} [-x_2(t)(2\beta_n x_2(t) + \sigma_n^2 x_1(t))(dt)^2 - \sigma_n^2 x_1(t)\, x_2(t)\, dt E\{dB(t)\}]$$

$$= 0 \qquad (4.18)$$

$$b_{22}(x,t) = \lim_{dt \to 0} \frac{1}{dt} E\{ [-(2\beta_n x_2(t) + \sigma_n^2 x_1(t))\,dt - \sigma_n^2 x_1(t)\,dB(t)]^2 \mid x(t) = x\}$$

$$= \lim_{dt \to 0} \frac{1}{dt} [(2\beta_n x_2(t) + \sigma_n^2 x_1(t))^2 (dt)^2 + 2(2\beta_n x_2(t)$$

$$+ \sigma_n^2 x_1(t))\,\sigma_n^2 x_1(t)\,dt\; E\{dB(t)\}$$

$$+ \sigma_n^4 x_1^2(t)\; E\{[dB(t)]^2\}]$$

$$= \lim_{dt \to 0} \frac{1}{dt} [\sigma_n^4 x_1^2(t)\, S_W dt] \qquad \text{(See equation (4.2).)}$$

$$= \sigma_n^4 S_W x_1^2(t) \qquad (4.19)$$

The Fokker Planck equation for the system (4.14) or (4.10) is obtained by substituting (4.15) through (4.19) into (4.11)

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x_2^2} [\sigma_n^4 S_W x_1^2 p] - \frac{\partial}{\partial x_1} [x_2 p] + \frac{\partial}{\partial x_2} [(2\beta_n x_2 + \sigma_n^2 x_1) p] \qquad (4.20)$$

We are interested in finding the second moments of the state vector components, i.e.,

$$m_{2,0} \equiv E\{x_1^2(t)\}$$

$$m_{1,1} \equiv E\{x_1(t)\,x_2(t)\}$$

$$m_{0,2} \equiv E\{x_2^2(t)\} \tag{4.21}$$

These can be determined directly by a formal procedure of multiplying the Fokker Planck equation (4.20) by

$$x_1^i\,x_2^j,\; i+j=2,$$

and integrating over all (x_1, x_2).

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial p}{\partial t}\,dx_1 dx_2 = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial^2}{\partial x_2^2}[\sigma_n^4 S_W x_1^2 p]\,dx_1 dx_2$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial}{\partial x_1}[x_2 p]\,dx_1 dx_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial}{\partial x_2}[(2\beta_n x_2 + \sigma_n^2 x_1)p]\,dx_1 dx_2 \tag{4.22}$$

But

$$\dot{m}_{i,j} = \frac{d}{dt}E\{x_1^i x_2^j\} = \frac{d}{dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j p\,dx_1 dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial p}{\partial t}\,dx_1 dx_2$$

which is just the left side of (4.22). Now integrate by parts with respect to x_1 the second term on the right of (4.22).

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial}{\partial x_1}[x_2 p]\, dx_1 dx_2 = \int_{-\infty}^{\infty} \left\{ x_1^i p \Big|_{-\infty}^{\infty} - i \int_{-\infty}^{\infty} x_1^{i-1} p\, dx_1 \right\} x_2^{j+1} dx_2 \qquad (4.23)$$

We will assume that the second moments exist which implies that the first term on the right of (4.23) vanishes. Hence (4.23) becomes:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^i x_2^j \frac{\partial}{\partial x_1}[x_2 p]\, dx_1 dx_2 = -i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^{i-1} x_2^{j+1} p\, dx_1 dx_2 = -iE\left\{x_1^{i-1} x_2^{j+1}\right\}$$

$$= -im_{i-1,j+1}, \qquad i+j=2 \qquad (4.24)$$

Evaluating the remaining two terms, (4.22) becomes

$$\dot{m}_{i,j} = \frac{1}{2}\sigma_n^4 S_W j(j-1) m_{i+2,j-2} + im_{i-1,j+1} - 2\beta_n j m_{i,j} - \sigma_n^2 j m_{i+1,j-1}, \qquad i+j=2 \qquad (4.25)$$

This gives the following set of coupled linear first order constant coefficient differential equations for the second moments of the state vector components.

$$\dot{m}_{2,0} = 2m_{1,1}$$

$$\dot{m}_{1,1} = m_{0,2} - 2\beta_n m_{1,1} - \sigma_n^2 m_{2,0}$$

$$\dot{m}_{0,2} = \sigma_n^4 S_W m_{2,0} - 2\sigma_n^2 m_{1,1} - 4\beta_n m_{0,2}$$

or in vector-matrix notation

$$\frac{d}{dt}\begin{bmatrix} m_{2,0} \\ m_{1,1} \\ m_{0,2} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -\sigma_n^2 & -2\beta_n & 1 \\ \sigma_n^4 S_W & -2\sigma_n^2 & -4\beta_n \end{bmatrix} \begin{bmatrix} m_{2,0} \\ m_{1,1} \\ m_{0,2} \end{bmatrix} \tag{4.26}$$

For (4.26) to be a stable system, its characteristic equation

$$\lambda^3 + 6\beta_n \lambda^2 + (8\beta_n^2 + 4\sigma_n^2)\lambda + 2\sigma_n^2 (4\beta_n - \sigma_n^2 S_W) = 0 \tag{4.27}$$

must have eigenvalues with negative real parts. The Routh Hurwitz criterion yields for stability

$$S_W < 4\frac{\beta_n}{\sigma_n^2} \tag{4.28}$$

To obtain (4.6) we have from (4.4)

$$E\{f_n^2(t)\} = \frac{4}{\ell^2}\int_0^\ell\int_0^\ell E\{w(x,t)\,w(x',t)\}\sin\frac{n\pi}{\ell}x\,\sin\frac{n\pi}{\ell}x'\,dx dx' \tag{4.29}$$

Assuming $E\{w(x,t)\,w(x',t)\}$ is bounded initially over 0 to ℓ, since $E\{f_n^2(t)\}$ is stable for all n by (4.28), and since

$$\int_0^\ell \sin\frac{n\pi}{\ell}x\,dx = \frac{\ell}{n\pi}\,[1-(-1)^n] = 0\,(\frac{1}{n})$$

then from (4.29) we have for any $t \geq 0$

$$E\{ f_n^2(t)\} = 0\left(\frac{1}{n^2}\right) = 0\left(\frac{1}{\omega_n^2}\right) \tag{4.30}$$

Since (4.28) assures that (4.27) will have eigenvalues with negative real parts for all n, then a $b > 0$, independent of n, can be found which is a lower bound for the absolute values of these real parts. Thus by multiplying (4.30) by ω_n^2 we see that an $a > 0$, independent of n, can be found such that (4.6) holds.

Denoting the power spectral density function of Φ (t) by S_Φ and using the notation of (2.10), (2.13), and (2.14), (4.28) becomes

$$S_\Phi < 4\beta_n EIm \left(1 - \frac{P_0}{P_n^*}\right) \tag{4.31}$$

Upon satisfaction of (4.31) for all n, the second moment of the rod deflection will be asymptotically stable. In fact, by definition 4.1 of section 4.2.2, it is exponentially stable. The results of an analog computer simulation of (4.8) for a wide band noise longitudinal force is included in section 4.2.2 where some sample solutions are exhibited.

4.1.2 Second Moment Stability for Narrow Band Excitation

In the previous section, the study of the stability of a simply supported rod subjected to a random longitudinal force led us to consider the equation

$$\ddot{f}_n + 2\beta_n \dot{f}_n + \sigma_n^2 [1 + \mu_n \Phi(t)] f_n = 0 \tag{4.32}$$

In that section we considered a white noise excitation and in section 4.1.3 we will consider $\Phi(t)$ to be a nonwhite physical noise.

Another case of interest occurs when the $\Phi(t)$ random function is a narrow-band noise. That is, most of the average power, as a function of the frequency content, is centered at a single frequency. One method of constructing a model of narrow band noise is to superimpose a small white noise on a cosine function. With this model, both Markov process theory and Floquet theory can be used, respectively, to derive moment equations with periodic coefficients and to study their stability.

The damping coefficients β_n for the vibration modes of the rod which exhibit the most response are usually quite small. A conservative assumption is to neglect damping. In this case (4.32) becomes

$$\ddot{f}_n + \sigma_n^2 [1 + \mu_n \Phi(t)] f_n = 0 \tag{4.33}$$

Now if $\Phi(t) = \cos\omega t$, then (4.33) is the Mathieu equation which admits bounded (i.e., stable) solutions in certain regions of the (σ_n^2, μ_n) plane. It is of interest to see if a similar result can be obtained when a small white noise is superimposed on the cosine function. That is, when

$$\Phi(t) = \cos\omega t + W(t) \tag{4.34}$$

where W(t) is the Gaussian white noise with mean zero and

$$E\left\{W(t)\, W(t+\tau)\right\} = S_W \delta(\tau) \tag{4.35}$$

Now, are there (σ_n^2, μ_n, S_W) regions for which the second moments of the solution process of (4.33) (or, better yet, the samples themselves) are bounded?

Remembering that the Gaussian white noise can be represented as the formal derivative of the Brownian motion process, $W(t) = \dot{B}(t)$, we can rewrite the equation (4.33), using (4.34), as the system of differentials

$$dx_1 = x_2 dt$$

$$dx_2 = -\sigma_n^2 (1+\mu_n \cos\omega t)\, x_1 dt - \mu_n \sigma_n^2 x_1 dB \qquad (4.36)$$

where

$$x_1 = f_n$$

$$x_2 = \dot{x}_1 \qquad (4.37)$$

Referring to (4.14 - 4.20) of section 4.1.1, we can write the Fokker-Planck equation for the density function of the system (4.36) as

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x_1}[x_2 p] + \frac{\partial}{\partial x_2}[\sigma_n^2 (1+\mu_n \cos\omega t)\, x_1 p] + \frac{1}{2}\frac{\partial^2}{\partial x_2^2}[(\mu_n \sigma_n^2)^2 S_W x_1^2 p]$$

$$= -x_2 \frac{\partial p}{\partial x_1} + \sigma_n^2 (1+\mu_n \cos\omega t)\, x_1 \frac{\partial p}{\partial x_2} + \frac{1}{2}(\mu_n \sigma_n^2)^2 S_W x_1^2 \frac{\partial^2 p}{\partial x_2^2} \qquad (4.38)$$

where

$$E\left\{(dB)^2\right\} = S_W dt.$$

Paralleling (4.21 - 4.26) of the same section, we arrive at the following set of differential equations for the second moments of the solution of (4.36).

$$\dot{m}_{2,0} = 2m_{1,1}$$

$$\dot{m}_{1,1} = -\sigma_n^2 (1+\mu_n \cos\omega t) m_{2,0} + m_{0,2} \qquad (4.39)$$

$$\dot{m}_{0,2} = (\mu_n \sigma_n^2)^2 S_W m_{2,0} - 2\sigma_n^2 (1+\mu_n \cos\omega t)\, m_{1,1}$$

The system of equations (4.39) is a multidimensional Mathieu system which can be put in vector-matrix form.

$$\frac{d}{dt}\begin{bmatrix} m_{2,0} \\ m_{1,1} \\ m_{0,2} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ -\sigma_n^2(1+\mu_n\cos\omega t) & 0 & 1 \\ (\mu_n\sigma_n^2)^2 S_W & -2\sigma_n^2(1+\mu_n\cos\omega t) & 0 \end{bmatrix}\begin{bmatrix} m_{2,0} \\ m_{1,1} \\ m_{0,2} \end{bmatrix} \qquad (4.40)$$

Since (4.40) is a system of differential equations with periodic coefficients, its stability properties can be investigated using Floquet theory. Instead of pursuing this we will obtain a qualitative result by considering the differential equation for

$$m_{2,0} = E\{x_1^2(t)\}.$$

Differentiating the first equation of (4.39) twice with respect to time and using the second and third equations gives

$$\frac{d^3}{dt^3}[m_{2,0}] = 2\frac{d^2}{dt^2}[m_{1,1}]$$

$$= -2\sigma_n^2\frac{d}{dt}[(1+\mu_n\cos\omega t)m_{2,0}] + 2\frac{d}{dt}[m_{0,2}]$$

$$= -4\sigma_n^2(1+\mu_n\cos\omega t)m_{1,1} - 2\sigma_n^2\frac{d}{dt}[(1+\mu_n\cos\omega t)m_{2,0}] + 2(\mu_n\sigma_n^2)^2 S_W m_{2,0}$$

or

$$\frac{d^3}{dt^3}[m_{2,0}] = -2\sigma_n^2(1+\mu_n\cos\omega t)\frac{d}{dt}[m_{2,0}] - 2\sigma_n^2\frac{d}{dt}[(1+\mu_n\cos\omega t)m_{2,0}] + 2(\mu_n\sigma_n^2)^2 S_W m_{2,0} \qquad (4.41)$$

To make a comparison of (4.41) with the deterministic equation, consider (4.33) and (4.34) without the white noise, i.e., $S_W = 0$,

$$\ddot{f}_n + \sigma_n^2(1+\mu_n\cos\omega t)f_n = 0 \qquad (4.42)$$

We are interested in expressing (4.42) as a differential equation in terms of the dependent variable f_n^2. First multiply (4.42) by $2\dot{f}_n$

$$2\dot{f}_n\ddot{f}_n + \sigma_n^2(1+\mu_n\cos\omega t)\,2f_n\dot{f}_n = 0$$

or

$$\frac{d}{dt}[\dot{f}_n^2] = -\sigma_n^2(1+\mu_n\cos\omega t)\frac{d}{dt}[f_n^2] \tag{4.43}$$

Now

$$\frac{d}{dt}[f_n^2] = 2f_n\dot{f}_n$$

$$\frac{d^2}{dt^2}[f_n^2] = 2\dot{f}_n^2 + 2f_n\ddot{f}_n \tag{4.44}$$

Substituting $\ddot{f}_n$ from (4.42) into (4.44) gives

$$\frac{d^2}{dt^2}[f_n^2] = 2\dot{f}_n^2 - 2\sigma_n^2(1+\mu_n\cos\omega t)f_n^2 \tag{4.45}$$

Differentiating (4.45) gives

$$\frac{d^3}{dt^3}[f_n^2] = 2\frac{d}{dt}[\dot{f}_n^2] - 2\sigma_n^2\frac{d}{dt}[(1+\mu_n\cos\omega t)f_n^2] \tag{4.46}$$

Substituting (4.43) into (4.46) gives

$$\frac{d^3}{dt^3}[f_n^2] = -2\sigma_n^2(1+\mu_n\cos\omega t)\frac{d}{dt}[f_n^2] - 2\sigma_n^2\frac{d}{dt}[(1+\mu_n\cos\omega t)f_n^2] \tag{4.47}$$

Compare equations (4.41) and (4.47). For $S_W = 0$, they are the same when f_n^2 is identified with $m_{2,0}$.

Since f_n satisfies the Mathieu equation (4.42), f_n^2 of (4.47) will possess the same stability regions which, in turn, will coincide with the stability regions of (4.41) for $S_W = 0$.

Now we will use a theorem similar to Theorem 4.1 of reference [18] on the continuous dependence of the solution of the differential equation (4.41) upon the parameter S_W. Since the right hand side of (4.41) is a continuous function of the argument S_W, and is bounded in finite regions of the state space, it follows that the solutions of (4.47), which coincide with those of (4.41) for $S_W = 0$, will be uniform limits on $[t_0, T]$ of the solutions of (4.41) as $S_W \to 0$.

Hence, for S_W small, the stability boundaries of the second moment $m_{2,0}$, satisfying (4.41), will coincide approximately with the stability boundaries of the Mathieu equation (4.42). Naturally, equation (4.40) can yield explicit results. But, on a qualitative basis, we can say that the narrow band noise obtained by summing a sinusoidal term and a small background wide band noise will not change the stability properties of the second moment of the rod deflection.

What can we say about the sample properties for this case? In a stable (σ_n^2, μ_n, S_W) region, the second moments are almost periodic as determined from Floquet theory. Thus the second moments are bounded. How is this reflected in the sample properties? All that can be stated at this time is that the bounded second moments guarantee that the sample solutions will grow no faster than $t^{1+\epsilon}$ where ϵ is any small positive constant. This is not too useful in studying the stability. We would like to say that almost all samples will be bounded or, even better, decay to zero. If there is damping in the system, say as given by (4.32), then it will follow from Floquet theory that the stable solutions to the second moment equation, which will now have a damping term present, will decay exponentially. In this case, as will be discussed in section 4.2.2, we can not only say that the sample solutions will be bounded, but in fact, they will also decay exponentially with probability one.

4.1.3 Power Spectral Density Criterion for Stability of the Second Moments

In the previous two sections, as will be done also in this section, stability is defined in terms of the moments of the solution of a second order differential equation. Ideally, all the finite dimensional probability distributions of the solution should be determined but in most cases this is impossible. The first order probability distribution gives a partial statistical description of the solution but in most cases of practical interest even this cannot be obtained. Some of the information contained in the first order probability distribution is given by the first and second moments and in the Gaussian case these moments completely describe this distribution. Due to the relative ease of determining moments, we will be satisfied with this partial description of the solution process.

When a linear differential equation has a random input without random coefficients, the solution can be expressed as the convolution of the impulse response and the input. Since the convolution integral is a linear operation on the input, the moments of the solution can be determined directly from the moments of the input by taking the expected value of powers of the convolution integral. For linear differential equations with white noise coefficients the Fokker Planck equation yields solution moments immediately but in the non-white noise case, to be considered here, there is no general method for obtaining solution moments.

For the first order differential equation with a continuous spectrum Gaussian process as the coefficient, the moments can be obtained exactly. The reason for this is the following. The solution of the first order differential equation, $\dot{x} = f(t)x$, $x(0) = 1$, can be written as the exponential of a definite integral of the coefficient, i.e., $x(t) = \exp[{}_0\int^t f(\tau)\, d\tau]$. But since the coefficient f(t) is Gaussian so is its definite integral and the expected value of the exponential of a Gaussian process can be determined exactly by using its characteristic function. This yields the moments of the solution.

For the second order differential equation with a Gaussian coefficient, the solution as a two component vector cannot be expressed as an exponential of a definite integral alone. See equations (4.60) and (4.61). Since the solution is a nonlinear function of the coefficient and is not the special case of an exponent of a Gaussian process, the moments cannot be determined exactly. However, with appropriate assumptions, this will be done approximately. This result is due to Graefe [11].

We are interested in the stability of the equation

$$\ddot{f}_n + 2\beta_n \dot{f}_n + \sigma_n^2 [1 + \epsilon\mu_n \Phi(t)] f_n = 0 \qquad (4.48)$$

where ϵ is a small positive parameter and $\Phi(t)$ is not white noise but is now a mean zero, stationary, Gaussian noise with a continuous power spectral density function. Approximate stability conditions are derived which yield the intuitively appealing result that the stability of (4.48) depends only on the value of the power spectral density of $\Phi(t)$ at twice the damped natural frequency. It is also interesting to find that for small damping the value of the power spectral density determining stability is the same as that found in section 4.1.1 when $\Phi(t)$ was white noise.

Substituting

$$2\beta_n = 2\zeta_n \sigma_n \tag{4.49}$$

$$f_n(t) = e^{-\zeta_n \sigma_n t} q_n(t) \tag{4.50}$$

into (4.48) gives

$$\ddot{q}_n + \nu_n^2 [1 + \epsilon \bar{\mu}_n \Phi(t)] q_n = 0 \tag{4.51}$$

where

$$\nu_n = \sigma_n \sqrt{1 - \zeta_n^2} \tag{4.52}$$

$$\bar{\mu}_n = \frac{\mu_n}{1 - \zeta_n^2} \tag{4.53}$$

Assuming the solution of (4.51) to have slowly varying amplitude and phase, i.e., be in the form

$$q_n(t) = r_n(t) \cos(\nu_n t + \theta_n(t)) \equiv r_n \cos \varphi_n \tag{4.54}$$

where

$$\varphi_n(t) = \nu_n t + \theta_n(t) \tag{4.55}$$

and imposing the condition

$$\dot{q}_n = -\nu_n r_n \sin\varphi_n \tag{4.56}$$

yields the equation

$$\dot{r}_n \cos\varphi_n - r_n \dot{\theta}_n \sin\varphi_n = 0 \tag{4.57}$$

Substituting (4.54) and (4.56) into (4.51) gives

$$\dot{r}_n \sin\varphi_n + \dot{\theta}_n r_n \cos\varphi_n = \nu_n \bar{\mu}_n \epsilon \Phi(t) r_n \cos\varphi_n \tag{4.58}$$

Solving (4.57) and (4.58) simultaneously

$$\begin{aligned} \frac{\dot{r}_n}{r_n} &= \frac{1}{2}\nu_n \bar{\mu}_n \epsilon \Phi(t) \sin 2\varphi_n \\ \dot{\theta}_n &= \frac{1}{2}\nu_n \bar{\mu}_n \epsilon \Phi(t) [1 + \cos 2\varphi_n] \end{aligned} \tag{4.59}$$

The solution of (4.59) is

$$\begin{aligned} r_n(t) &= r_n(0) e^{A_n(t)} \\ \theta_n(t) &= \theta_n(0) + B_n(t) \end{aligned} \tag{4.60}$$

where

$$\begin{aligned} A_n(t) &= \frac{1}{2}\nu_n \bar{\mu}_n \epsilon \int_0^t \Phi(\tau) \sin 2\varphi_n(\tau)\, d\tau \\ B_n(t) &= \frac{1}{2}\nu_n \bar{\mu}_n \epsilon \int_0^t \Phi(\tau) [1 + \cos 2\varphi_n(\tau)]\, d\tau \end{aligned} \tag{4.61}$$

Note that the form of the solution (4.60) and (4.61) is similar to the exponential form of the solution to the first order differential equation with the exception that the exponent is not simply the definite integral of the random coefficient $\Phi(t)$.

To obtain an approximate stability condition we expand A_n and B_n in a power series of the small parameter ϵ

$$\begin{aligned} A_n(t) &= \epsilon A_n^{(1)}(t) + \epsilon^2 A_n^{(2)}(t) + \epsilon^3 A_n^{(3)}(t) + \cdots \\ B_n(t) &= \epsilon B_n^{(1)}(t) + \epsilon^2 B_n^{(2)}(t) + \epsilon^3 B_n^{(3)}(t) + \cdots \end{aligned} \tag{4.62}$$

and letting

$$\varphi_{n0} = \nu_n t + \theta_{n0} \tag{4.63}$$

we expand $\cos 2\varphi_n$ and $\sin 2\varphi_n$ in a Taylor series about $2\varphi_{n0}$

$$\begin{aligned}\cos 2\varphi_n &= \cos(2\varphi_{n0} + 2B_n) \\ &= \cos 2\varphi_{n0} - 2\epsilon B_n^{(1)} \sin 2\varphi_{n0} + O(\epsilon^2)\end{aligned} \tag{4.64}$$

$$\begin{aligned}\sin 2\varphi_n &= \sin(2\varphi_{n0} + 2B_n) \\ &= \sin 2\varphi_{n0} + 2\epsilon B_n^{(1)} \cos 2\varphi_{n0} + O(\epsilon^2)\end{aligned} \tag{4.65}$$

where the notation $O(\epsilon^2)$ represents terms with powers of $\epsilon \geq 2$. Substituting (4.62), (4.64), and (4.65) into (4.61) and equating coefficients of ϵ and ϵ^2 gives

$$\begin{aligned}
A_n^{(1)}(t) &= \frac{1}{2}\nu_n \bar{\mu}_n \int_0^t \Phi(\tau) \sin 2\varphi_{n0}(\tau)\, d\tau \\
A_n^{(2)}(t) &= \nu_n \bar{\mu}_n \int_0^t \Phi(\tau) \cos 2\varphi_{n0}(\tau)\, B_n^{(1)}(\tau)\, d\tau \\
B_n^{(1)}(t) &= \frac{1}{2}\nu_n \bar{\mu}_n \int_0^t \Phi(\tau)\, [1 + \cos 2\varphi_{n0}(\tau)]\, d\tau \\
B_n^{(2)}(t) &= -\nu_n \bar{\mu}_n \int_0^t \Phi(\tau) \sin 2\varphi_{n0}(\tau)\, B_n^{(1)}(\tau)\, d\tau
\end{aligned} \tag{4.66}$$

To obtain a condition for asymptotic stability of the second moment of the rod deflection we follow exactly the same argument as in section 4.1.1 equations (4.4) through (4.7). From (4.50), (4.54), and (4.60)

$$\begin{aligned}
E\left\{f_n^2(t)\right\} &= r_n^2(0)\, e^{-2\zeta_n \sigma_n t}\, E\left\{e^{2A_n(t)} \cos^2 \varphi_n(t)\right\} \\
&= \frac{1}{2} r_n^2(0)\, e^{-2\zeta_n \sigma_n t}\Big[E\left\{e^{2A_n(t)}\right\} + \cos 2\varphi_{n0}(t)\, E\left\{e^{2A_n(t)} \cos 2B_n(t)\right\} \\
&\quad - \sin 2\varphi_{n0}(t)\, E\left\{e^{2A_n(t)} \sin 2B_n(t)\right\}\Big]
\end{aligned} \tag{4.67}$$

Since $\Phi(t)$ is Gaussian and linear operations on Gaussian processes yield Gaussian processes, $A_n^{(1)}(t)$ and $B_n^{(1)}(t)$ are Gaussian but $A_n^{(2)}(t)$ and $B_n^{(2)}(t)$ are not. Thus $A_n(t)$ and $B_n(t)$ are only approximately Gaussian. To determine the expected values in (4.67), the joint density function of $A_n(t)$ and $B_n(t)$ is required. We will assume that $A_n(t)$ and $B_n(t)$ are jointly Gaussian which leads to the approximate stability condition (4.69) below. If the joint density function of $A_n^{(1)}$, $A_n^{(2)}$, $B_n^{(1)}$, and $B_n^{(2)}$ could be determined, then (4.68) and therefore the stability condition (4.69) would include the various moments of $A_n^{(1)}$, $A_n^{(2)}$, $B_n^{(1)}$, $B_n^{(2)}$ and not simply those of A_n and B_n.

Assuming A_n and B_n to be jointly Gaussian, the expected values in (4.67) can be determined from the characteristic function or by direct integration.

$$
\begin{aligned}
E\left\{e^{2A_n(t)}\right\} &= e^{2(m_{A_n}+\gamma_{A_n^2})} \\
E\left\{e^{2A_n(t)}\cos 2B_n(t)\right\} &= \cos 2(m_{B_n}+2\gamma_{A_nB_n})\, e^{2(m_{A_n}+\gamma_{A_n^2}-\gamma_{B_n^2})} \\
E\left\{e^{2A_n(t)}\sin 2B_n(t)\right\} &= \sin 2(m_{B_n}+2\gamma_{A_nB_n})\, e^{2(m_{A_n}+\gamma_{A_n^2}-\gamma_{B_n^2})}
\end{aligned}
\tag{4.68}
$$

where

$$
\begin{aligned}
m_{A_n} &= E\left\{A_n(t)\right\} & \gamma_{A_n^2} &= E\left\{(A_n - m_{A_n})^2\right\} \\
m_{B_n} &= E\left\{B_n(t)\right\} & \gamma_{B_n^2} &= E\left\{(B_n - m_{B_n})^2\right\} \\
& & \gamma_{A_nB_n} &= E\left\{(A_n - m_{A_n})(B_n - m_{B_n})\right\}
\end{aligned}
$$

Substituting (4.68) into (4.67)

$$
E\left\{f_n^2(t)\right\} = \frac{1}{2} r_n^2(0)\, e^{2(-\zeta_n\sigma_n t + m_{A_n} + \gamma_{A_n^2})}\Big[1 + \cos 2\varphi_{n0}\cos 2(m_{B_n} + 2\gamma_{A_nB_n})\, e^{-2\gamma_{B_n^2}} - \sin 2\varphi_{n0}\sin 2(m_{B_n} + 2\gamma_{A_nB_n})\, e^{-2\gamma_{B_n^2}}\Big]
$$

from which the condition for asymptotic stability is

$$-\zeta_n \sigma_n + \lim_{t\to\infty} \frac{1}{t}\left[m_{A_n} + \gamma_{A_n^2}\right] < 0 \tag{4.69}$$

Since $E\{\Phi(t)\} = 0$, then $E\{A_n^{(1)}(t)\} = 0$ and

$$m_{A_n} = \epsilon^2 E\left\{A_n^{(2)}\right\} + 0(\epsilon^3)$$

$$\gamma_{A_n^2} = \epsilon^2 E\left\{\left(A_n^{(1)}\right)^2\right\} + 0(\epsilon^3)$$

Retaining terms of the order of smallness up to ϵ^2, (4.69) becomes

$$-\zeta_n \sigma_n + \lim_{t\to\infty} \frac{\epsilon^2}{t}\left[E\left\{A_n^{(2)}\right\} + E\left\{\left(A_n^{(1)}\right)^2\right\}\right] < 0 \tag{4.70}$$

From (4.66)

$$E\left\{A_n^{(2)}\right\} = \frac{1}{4}(\nu_n\bar{\mu}_n)^2 \int_0^t\int_0^{\tau_2} \Gamma_\Phi(\tau_1-\tau_2)\left\{2\cos 2(\nu_n\tau_2+\theta_{n0}) + \cos[2\nu_n(\tau_1+\tau_2)+4\theta_{n0}] + \cos 2\nu_n(\tau_1-\tau_2)\right\} d\tau_1\, d\tau_2 \tag{4.71}$$

$$E\left\{\left(A_n^{(1)}\right)^2\right\} = \frac{1}{8}(\nu_n\bar{\mu}_n)^2 \int_0^t\int_0^t \Gamma_\Phi(\tau_1-\tau_2)\left\{\cos 2\nu_n(\tau_1-\tau_2) - \cos[2\nu_n(\tau_1+\tau_2)+4\theta_{n0}]\right\} d\tau_1\, d\tau_2 \tag{4.72}$$

where

$$\Gamma_\Phi(\tau) = E\left\{\Phi(t)\Phi(t+\tau)\right\}$$

is the covariance function of the zero mean process $\{\Phi(t),\ t\epsilon[0,\infty)\}$. The covariance and the power spectral density functions are Fourier transform pairs, i.e.,

$$S_\Phi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\,\Gamma_\Phi(\tau)\, d\tau$$

$$\Gamma_\Phi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau}\, S_\Phi(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\omega\tau\, S_\Phi(\omega)\, d\omega \tag{4.73}$$

Substituting (4.73) into (4.71) and (4.72), and assuming $S_\Phi(\omega)$ converges to zero fast enough as $\omega\to\infty$ so that the order of integration can be interchanged, yields the following five integrals to be evaluated.

$$I_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t\int_0^{\tau_2} S_\Phi(\omega)\cos\omega(\tau_1-\tau_2)\cos(2\nu_n\tau_2+2\theta_{n0})\,d\tau_1\,d\tau_2\,d\omega$$

$$I_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t\int_0^{\tau_2} S_\Phi(\omega)\cos\omega(\tau_1-\tau_2)\cos 2\nu_n(\tau_1-\tau_2)\,d\tau_1\,d\tau_2\,d\omega$$

$$I_3 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t\int_0^{\tau_2} S_\Phi(\omega)\cos\omega(\tau_1-\tau_2)\cos[2\nu_n(\tau_1+\tau_2)+4\theta_{n0}]\,d\tau_1\,d\tau_2\,d\omega \qquad (4.74)$$

$$I_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t\int_0^{t} S_\Phi(\omega)\cos\omega(\tau_1-\tau_2)\cos 2\nu_n(\tau_1-\tau_2)\,d\tau_1\,d\tau_2\,d\omega = 2I_2$$

$$I_5 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t\int_0^{t} S_\Phi(\omega)\cos\omega(\tau_1-\tau_2)\cos[2\nu_n(\tau_1+\tau_2)+4\theta_{n0}]\,d\tau_1\,d\tau_2\,d\omega = 2I_3$$

Using the following relation from [19].

$$\lim_{t\to\infty}\left[\frac{1}{2\pi t}\,\frac{\sin^2\omega t}{\omega^2}\right] = \delta(2\omega)$$

and the fact that $S_\Phi(\omega)$ is a continuous even function of ω, [11] evaluates the five integrals (4.74) and obtains

$$\lim_{t\to\infty}\frac{1}{t}I_1 = \lim_{t\to\infty}\frac{1}{t}I_3 = \lim_{t\to\infty}\frac{1}{t}I_5 = 0 \qquad (4.75)$$

$$\lim_{t\to\infty}\frac{1}{t}I_2 = \frac{1}{4}\int_{-\infty}^{\infty} S_\Phi(\omega)\lim_{t\to\infty}\frac{1}{2\pi t}\left[\frac{\sin^2\left(\frac{\omega+2\nu_n}{2}\right)t}{\left(\frac{\omega+2\nu_n}{2}\right)^2}+\frac{\sin^2\left(\frac{\omega-2\nu_n}{2}\right)t}{\left(\frac{\omega-2\nu_n}{2}\right)^2}\right]d\omega$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} S_\Phi(\omega)\left[\delta(\omega+2\nu_n)+\delta(\omega-2\nu_n)\right]d\omega$$

$$= \frac{1}{2}S_\Phi(2\nu_n) \qquad (4.76)$$

and from (4.74)

$$\lim_{t\to\infty} \frac{1}{t} I_4 = \lim_{t\to\infty} \frac{2}{t} I_2 = S_\Phi\,(2\nu_n) \tag{4.77}$$

Substituting (4.75), (4.76), and (4.77) into (4.70) gives the stability condition

$$-\zeta_n\sigma_n + \frac{\varepsilon^2}{4}\,(\nu_n\bar{\mu}_n)^2\, S_\Phi\,(2\nu_n) < 0 \tag{4.78}$$

which when satisfied for all integers n implies that the second moment of the rod deflection will asymptotically converge to zero as $t\to\infty$. Notice that the power spectral density is evaluated at only one frequency, i.e., twice the damped natural frequency of (4.48).

Using the original terminology, i.e.,

$$\beta_n = \zeta_n\sigma_n,\ \nu_n = \sigma_n\sqrt{1-\zeta_n^2}\ ,\ \bar{\mu}_n = \mu_n/(1-\zeta_n^2)$$

(4.78) becomes

$$\varepsilon^2 S_\Phi\,(2\nu_n) < \frac{4\beta_n(\sigma_n^2-\beta_n^2)}{\sigma_n^4\mu_n^2} \qquad \text{for all n} \tag{4.79}$$

which for low damping values $\beta_n^2 << \sigma_n^2$ becomes approximately

$$\varepsilon^2 S_\Phi\,(2\sigma_n) < \frac{4\beta_n}{\sigma_n^2\mu_n^2}$$

$$= 4\beta_n \mathrm{EIm}(1 - \frac{P_o}{P_n^*}) \qquad \text{for all n} \tag{4.80}$$

In the stability condition for low damping (4.80), the power spectral density level on the stability boundary is the same as that in (4.31) of section 4.1.1 for white noise; however, since (4.80) is an approximate stability condition and the physical interpretation of the white noise result is open to question, this can only be considered as an interesting coincidence at this time.

4.2 Conditions for Sample Stability

4.2.1 Sufficient Conditions for Asymptotic Sample Stability

The methods presented in the previous sections were primarily concerned with stability of the rod deflection in an average sense, i.e., stability of the second moments. It seems obvious that the most desirable result would be the stability of the sample solutions for the rod deflection. The approach presented here is due to Infante [14] and yields a sufficient condition for asymptotic sample stability when the longitudinal force is a stationary ergodic stochastic process. It is important to note that this result is not valid for the Gaussian white noise since the calculus required is not used.

As before, we consider the following infinite series for the rod deflection

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{\ell} x \, f_n(t) \tag{4.81}$$

Since the equations involved are linear, asymptotic sample stability requires that

$$\lim_{t\to\infty} w(x,t) = 0 \tag{4.82}$$

with probability one.

Assuming that the $f_n(t)$ are asymptotically stable (the conditions for this will be determined in this section) then they will be sample bounded over t. Using the Weierstrass M-test, we obtain the uniform convergence of (4.81) with respect to t and can then interchange the limit and summation.

Thus

$$\begin{aligned} \lim_{t\to\infty} w(x,t) &= \lim_{t\to\infty} \sum_{n=1}^{\infty} \sin \frac{n\pi}{\ell} x \, f_n(t) \\ &= \sum_{n=1}^{\infty} \sin \frac{n\pi}{\ell} x \lim_{t\to\infty} f_n(t) \\ &= 0 \end{aligned} \tag{4.83}$$

will be satisfied if

$$\lim_{t\to\infty} f_n(t) = 0 \qquad \text{for all } n \tag{4.84}$$

The differential equation for $f_n(t)$, (2.12), is

$$\ddot{f}_n(t) + 2\beta_n \dot{f}_n(t) + \sigma_n^2 [1 + \mu_n \Phi(t)] f_n(t) = 0 \tag{4.85}$$

Letting $\mu\Phi(t) \equiv f(t)$ be a mean zero stationary ergodic stochastic process, writing (4.85) in state vector notation, and suppressing the subscript n on the components of the state vector, i.e., $x_1 = f_n$, $x_2 = \dot{x}_1$, gives

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2\beta_n x_2 - \sigma_n^2 [1 + f(t)] x_1 \tag{4.86}$$

or

$$\dot{x} = [A + F(t)] x \tag{4.87}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -\sigma_n^2 & -2\beta_n \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 & 0 \\ -\sigma_n^2 f & 0 \end{bmatrix} \tag{4.88}$$

The sufficient condition for sample stability of (4.85) is obtained by defining a Liapunov function which is a positive definite quadratic form in the components of x and then determining conditions which ensure that this function, and therefore x, approaches zero as $t \to \infty$.

To determine these conditions, the following result from linear algebra is required [20] Let D and B be nxn real symmetric matrices with B positive definite. The form $x'Dx - x'Bx$ is referred to in linear algebra texts (see [20] for instance) as a pencil. Its characteristic equation is defined to be $|D - \lambda B| = 0$ and the λ which satisfy this are the eigenvalues of the pencil. Denote the eigenvalues of an nxn matrix Q as $\lambda_1[Q], \ldots, \lambda_n[Q]$ and the maximum and minimum eigenvalues as $\lambda_{max}[Q]$, and $\lambda_{min}[Q]$, respectively. The characteristic equation of the pencil $x'Dx - x'Bx$ has n real eigenvalues λ_s. The matrix DB^{-1} has the same eigenvalues as the pencil and

$$\lambda_{min}[DB^{-1}] = \min_x \frac{x'Dx}{x'Bx}$$

$$\lambda_{max}[DB^{-1}] = \max_x \frac{x'Dx}{x'Bx} \tag{4.89}$$

Now for the Liapunov function consider the positive definite quadratic form

$$V(x) = x' Bx \tag{4.90}$$

Evaluating the time derivative of (4.90) along the trajectories of (4.87) gives

$$\dot{V}(x) = \dot{x}' Bx + x' B\dot{x} = x' [(A + F)' B + B (A + F)] x \tag{4.91}$$

Define the function

$$\lambda (t) = \frac{\dot{V}(x)}{V(x)} = \frac{x' [(A + F)' B + B (A + F)] x}{x' Bx} \tag{4.92}$$

Identifying $(A + F)' B + B (A + F)$ in (4.92) with D in (4.89) yields

$$\lambda_{min} [(A + F)' + B(A + F)B^{-1}] \leq \lambda (t) \leq \lambda_{max} [(A + F)' + B(A + F)B^{-1}] \tag{4.93}$$

Integrating the first order differential equation (4.92) gives

$$V[x(t)] = V[x(t_0)] e^{\int_{t_0}^{t} \lambda (\tau) d\tau} = V[x(t_0)] e^{[\frac{1}{t-t_0} \int_{t_0}^{t} \lambda (\tau) d\tau](t-t_0)} \tag{4.94}$$

Invoking the ergodic hypothesis, i.e.,

$$\lim_{t \to \infty} \frac{1}{t-t_0} \int_{t_0}^{t} \lambda (\tau) \, d\tau = E \{\lambda (t)\}$$

we write (4.94) as

$$\lim_{t \to \infty} V [x(t)] = V [x(t_0)] e^{\lim_{t \to \infty} \left[E \{\lambda(t)\} (t-t_0)\right]} \tag{4.95}$$

From this it follows that if $E\{\lambda (t)\} \leq - \epsilon$ for some $\epsilon > 0$ then $V[x(t)]$ is bounded and converges to zero as $t \to \infty$. Since $V(x) = x' Bx$ and B is positive definite, then $V(x) > 0$ for all $x \neq 0$ and $V(x) = 0$ for $x = 0$. This implies that x will approach zero as $V(x)$ does.

From the inequality (4.93) we can ensure that $E\{\lambda (t)\} \leq - \epsilon$ if we specify

$$E\{\lambda_{max}[(A + F)' + B(A + F)B^{-1}]\} \leq - \epsilon, \quad \epsilon > 0 \tag{4.96}$$

This then is a sufficient condition for asymptotic sample stability of the system of differential equations (4.87). It is necessary that A have eigenvalues with negative real parts, which corresponds to $\beta_n > 0$, for (4.96) to hold.

To obtain the best sufficient condition for the system (4.87) and the particular Liapunov function chosen we specify the positive definite and symmetric B matrix in terms of two parameters α_1 and α_2. These parameters will then be determined by maximizing $E\{f^2(t)\}$.

$$B = \begin{bmatrix} \alpha_1^2 + \alpha_2 & \alpha_1 \\ \alpha_1 & 1 \end{bmatrix}, \quad \alpha_2 > 0 \tag{4.97}$$

Its inverse is

$$B^{-1} = \frac{1}{\alpha_2} \begin{bmatrix} 1 & -\alpha_1 \\ -\alpha_1 & \alpha_1^2 + \alpha_2 \end{bmatrix} \tag{4.98}$$

The matrix occurring in (4.96) and its maximum eigenvalue are

$$A' + F' + B[A + F]B^{-1} =$$

$$\frac{1}{\alpha_2} \begin{bmatrix} -\alpha_1\sigma_n^2(1 + f)-\alpha_1^2(\alpha_1-2\beta_n)-\alpha_1\alpha_2 & \sigma_n^2(\alpha_1^2-\alpha_2)(1 + f) + (\alpha_1^2 + \alpha_2)[\alpha_1(\alpha_1-2\beta_n) + \alpha_2] \\ -\sigma_n^2(1 + f)-\alpha_1(\alpha_1-2\beta_n) + \alpha_2 & \alpha_1\sigma_n^2(1 + f) + (\alpha_1^2 + \alpha_2)(\alpha_1-2\beta_n)-2\beta_n\alpha_2 \end{bmatrix} \tag{4.99}$$

$$\lambda_{max} = -2\beta_n + \left\{ 4(\beta_n-\alpha_1)^2 + \frac{1}{\alpha_2}[-\sigma_n^2(1 + f) + \alpha_2 + \alpha_1^2 + 2\alpha_1(\beta_n-\alpha_1)]^2 \right\}^{1/2} \tag{4.100}$$

Substituting (4.100) into (4.96) gives

$$E[\{4(\beta_n-\alpha_1)^2 + \frac{1}{\alpha_2}[-\sigma_n^2(1 + f) + \alpha_2 + \alpha_1^2 + 2\alpha_1(\beta_n-\alpha_1)]^2\}^{1/2}] \leq 2\beta_n-\epsilon \tag{4.101}$$

Using the Schwarz inequality we get a sufficient condition implying the inequality (4.101)

$$0 \le \left[E\left\{4(\beta_n-\alpha_1)^2 + \frac{1}{\alpha_2}[-\sigma_n^2(1+f) + \alpha_2 + \alpha_1^2 + 2\alpha_1(\beta_n-\alpha_1)]^2\right\}\right]^{1/2} \le 2\beta_n - \epsilon$$

which, after squaring both sides and using $E\{f(t)\} = 0$, becomes

$$\sigma_n^4 E\{f^2(t)\} \le 4\alpha_1\alpha_2(2\beta_n - \alpha_1) - [-\sigma_n^2 + \alpha_2 + \alpha_1^2 + 2\alpha_1(\beta_n - \alpha_1)]^2 - \epsilon' \qquad (4.102)$$

To determine α_1 and α_2 we define

$$G(\alpha_1, \alpha_2) = 4\alpha_1\alpha_2(2\beta_n - \alpha_1) - [-\sigma_n^2 + \alpha_2 + \alpha_1^2 + 2\alpha_1(\beta_n - \alpha_1)]^2$$

and find the maximum of the right side of (4.102) by

$$\frac{\partial G}{\partial \alpha_1} = 4(\beta_n - \alpha_1)\,[\alpha_2 + \sigma_n^2 - \alpha_1(2\beta_n - \alpha_1)] = 0$$

$$\frac{\partial G}{\partial \alpha_2} = 2\,[-\alpha_2 + \sigma_n^2 + \alpha_1(2\beta_n - \alpha_1)] = 0$$

which yields the optimal values of α_1 and α_2

$$\alpha_1 = \beta_n$$

$$\alpha_2 = \beta_n^2 + \sigma_n^2 \qquad (4.103)$$

Substituting the values (4.103) into (4.102) gives the desired sufficient condition for the asymptotic sample stability of the rod deflection

$$E\{f^2(t)\} < \frac{4\beta_n^2}{\sigma_n^2} \qquad \text{for all } n \qquad (4.104)$$

Using the relations $\mu_n \Phi(t) = f(t)$, $\mu_n = 1/(P_o - P_n^*)$, and $\sigma_n = \omega_n \sqrt{1-P_o/P_n^*}$, (4.104) becomes

$$E\left\{\Phi^2(t)\right\} < \frac{4\beta_n^2}{\sigma_n^2 \mu_n^2}$$

$$= 4\beta_n^2 \; EIm\left(1 - \frac{P_o}{P_n^*}\right) \quad \text{for all n.} \tag{4.105}$$

The criterion (4.105) for the asymptotic sample stability of the rod deflection is a condition on the variance of $\Phi(t)$, the zero mean portion of the longitudinal force, whereas the stability criteria in sections 4.1.1, 2, and 3 are in terms of the power spectral density of $\Phi(t)$.

Equation (4.105) is displayed in figure 4-1 where the stable region is shaded.

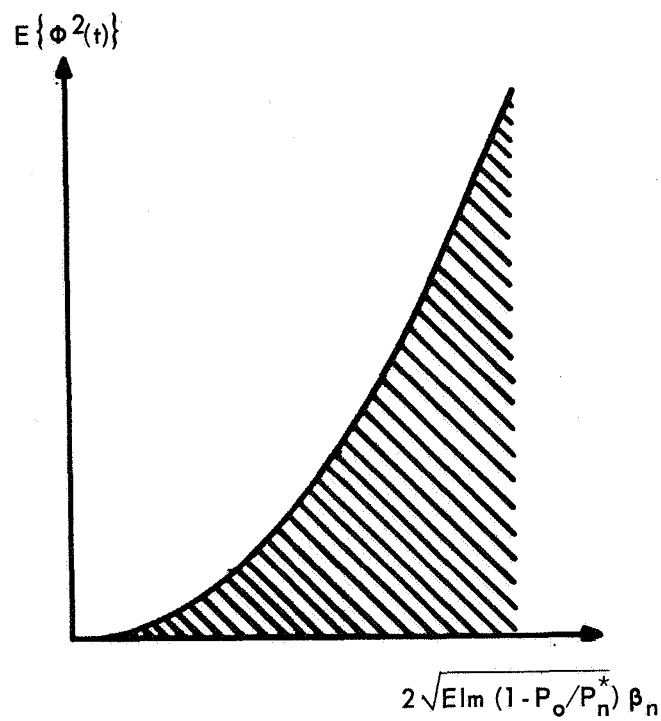

Figure 4-1. Sufficient Sample Stability Boundary for Equation (4.85)

4.2.2 Conditions for Exponential Stability of the Second Moments to Imply Asymptotic Sample Stability

In this section we present and discuss a theorem which relates moment stability to sample stability of the solution process, $\{x(t; x_0, t_0),\ t \in [t_0, \infty)\}$, of the following linear system

$$\dot{x} = F(t)x, \quad x(t_0) = x_0 \tag{4.106}$$

where x is an n-vector and F(t) is a matrix with components that are constants and/or stochastic processes.

The theorem concerns processes that are not white noise. A parallel theorem with essentially the same results for the white noise case is presented in [21] and proved in [10].

Definition 4.1. Exponential Stability of the Second Moments for Linear Stochastic Systems

The equilibrium state solution $x \equiv 0$ of the linear stochastic system (4.106) possesses exponential stability of the second moments if for any x_0 and $t \geq t_0$ there exists α, $\beta > 0$ such that

$$E\{\|x(t;x_0,t_0)\|^2\} \leq \alpha \|x_0\|^2 \exp[-\beta(t-t_0)] \tag{4.107}$$

where

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

In a considerable amount of the engineering literature devoted to stochastic stability, the concept of mean square stability has been used [3-5]. A system is mean square stable if for any x_0 there exists some $K > 0$ such that $\lim_{t\to\infty} E\{\|x(t;x_0,t_0)\|^2\} < K$. Obviously, a system with exponentially stable second moments will be mean square stable.

Theorem 4.1. Let the solution process $\{x(t;x_0, t_0), t \in [t_0, \infty)\}$ of the linear system (4.106) possess second moments that satisfy definition 4.1. That is, there exists α, $\beta > 0$ such that (4.107) holds for any x_0 and $t \geq t_0$. If the stochastic processes in F(t) are stationary and have finite second moments, then the equilibrium state solution is asymptotically sample stable in the sense of definition 3.6.

The proof of this theorem is given in [21] and will not be included here.

Note what this theorem says. Even though stability implications between moment properties and sample properties are generally invalid, the exponential stability of the second moments is a sufficient condition for the asymptotic stability of the samples. Since the method of obtaining asymptotic moment stability in sections 4.1.1, 4.1.2, and 4.1.3 was to bound the second moment of the solution process by a decaying exponential, then by Theorem 4.1, all the stability criteria obtained in those sections imply asymptotic sample stability.

Since Theorem 4.1 gives only a sufficient condition for asymptotic sample stability, the resulting stability criteria may be conservative. An illustration of the conservativeness of this condition for a first order differential equation is given in appendix A.

The analysis of section 4.1.1 and the result of Theorem 4.1 were verified experimentally on an EAI 221 analog computer. The differential equation

$$\ddot{f}_n(t) + 2\beta_n \dot{f}_n(t) + \sigma_n^2 [1 + \overline{W}(t)] f_n(t) = 0 \tag{4.108}$$

with the physical parameters

$$\begin{aligned} \beta_n &= 0.02 \text{ sec}^{-1} \\ \sigma_n &= 1.0 \text{ sec}^{-1} \\ f_n(0) &= 1.0 \text{ N.D.} \\ \dot{f}_n(0) &= 0.0 \text{ sec}^{-1} \end{aligned} \tag{4.109}$$

was simulated; the analog computer circuit diagram is shown in figure 4-9. The noise coefficient $\overline{W}$(t) was obtained from an Elgenco Model 311A Gaussian noise generator which has a power spectral density which is essentially constant from 0 to 40 c.p.s. and decreases rapidly for larger frequencies. With respect to the natural frequency 1 rad/sec of the differential equation (4.108), this represents a wide band noise approximation to a white noise coefficient.

The sufficient condition for asymptotic sample stability of (4.108) when the noise coefficient is white noise W(t) is

$$S_W < \frac{4\beta_n}{\sigma_n^2} \tag{4.110}$$

where S_W is the white noise power spectral density level. We will assume that the power spectral density $S_{\overline{W}}(\omega)$ of $\overline{W}(t)$ has the form

$$S_{\overline{W}}(\omega) = \begin{cases} S_{\overline{W}} & 0 \le |\omega| \le 2\pi x 40 \\ \\ 0 & |\omega| > 2\pi x 40, \end{cases} \tag{4.111}$$

that is, $S_{\overline{W}}(\omega)$ is constant for $0 \le |\omega| \le 2\pi x 40$ and zero elsewhere. The variance $\sigma_{\overline{W}}^2$ of $\overline{W}(t)$ can be computed from $S_{\overline{W}}(\omega)$ by

$$\begin{aligned} \sigma_{\overline{W}}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\overline{W}}(\omega)\, d\omega \\ &= \frac{2S_{\overline{W}}}{2\pi} \int_0^{2\pi x 40} d\omega \\ &= 80\, S_{\overline{W}} \end{aligned} \tag{4.112}$$

If we assume that we can use the wide band noise power spectral density level $S_{\overline{W}}$ in (4.112) for the white noise power spectral density level S_W in (4.110), then an approximate stability criterion in terms of $\sigma_{\overline{W}}^2$ for the asymptotic sample stability of (4.108) is

$$\begin{aligned} \sigma_{\overline{W}}^2 &< \frac{(80)(4\beta_n)}{\sigma_n^2} \\ &= \frac{(80)(.08)}{1.0} \\ &= 6.4 \text{ N.D.} \end{aligned} \tag{4.113}$$

This stability criterion is meaningful since for a wide band noise process $\sigma_{\overline{W}}^2 < \infty$ whereas (4.113) would be useless for the white noise $W(t)$ for which $\sigma_{W}^2 = \infty$.

The sample variance $\hat{\sigma}_{\overline{W}}^2$ was determined for each sample of the process $\{\overline{W}(t),\ t \in [0, T]\}$ from

$$\hat{\sigma}_{\overline{W}}^2 = \frac{1}{T} \int_0^T \overline{W}^2(t)\, dt \tag{4.114}$$

where T = 20 sec. was used for the simulation. We have used an ergodic hypothesis here to assure that the sample variance $\hat{\sigma}^2_W$ converges in some sense to the variance σ^2_W as $T \to \infty$.

Approximately 50 sample solutions of (4.108) were obtained for sample variances $\hat{\sigma}^2_W$ in the neighborhood of the values 0, 3, 6, 12, 20, and 30. Seven typical sample solutions for sample variances around those listed above are shown in figures 4-2 through 4-8. Notice the recorder amplitude scale change between figures 4-2 to 4-4 and figures 4-5 to 4-8. Figure 4-2 is included for comparison and shows the sample solution of (4.108) for $\hat{\sigma}^2_W = 0$ (no noise) and is simply the response of a lightly damped oscillatory second order system with the initial conditions $f_n(0) = 1.0$, $\dot{f}_n(0) = 0$. The remaining six figures are sample solutions which correspond respectively to the sample variances $\hat{\sigma}^2_W = 2.61, 5.72, 11.44, 12.01, 21.27$, and 29.98.

Due to saturation (overload) of the analog computer amplifiers, it is impossible to determine a precise stability boundary by analog simulation although the following statements can be made. For all sample variances $\hat{\sigma}^2_W < 6.0$, the sample solutions were stable (see, for example, figures 4-3 and 4-4). For all sample variances $\hat{\sigma}^2_W > 20.0$ the sample solutions were definitely unstable (see figures 4-7 and 4-8). Two sample solutions for sample variances in the neighborhood of $\hat{\sigma}^2_W = 12.0$ are shown in figures 4-5 and 4-6. These two solutions have roughly the same behavior up to the time that saturation occurred in figure 4-6, i.e., a slow growth in the amplitude of the oscillation. However, the sample solution in figure 4-5 did not exceed the saturation limit of the amplifiers and its amplitude increased and decreased randomly. This behavior was characteristic of all sample solutions with $\hat{\sigma}^2_W$ in the neighborhood of 12. Since the samples can be recorded for only a finite time, it is impossible to say whether either sample solution in figure 4-5 or 4-6 is stable or unstable. Intuitively, this indicates that the stability boundary is around $\hat{\sigma}^2_W = 12$.

Based on the approximate stability criterion (4.113) and the conclusion above, the stability criterion (4.110) is conservative.

That (4.110) is a sufficient condition for asymptotic sample stability is evident from figures 4-3 and 4-4. Thus, the analog computer simulation has verified the conclusion of Theorem 4.1. The interesting, and elusive, result is the necessary and sufficient condition for asymptotic sample stability of (4.108) when the noise coefficient is the Gaussian white noise. At the present time, this has not been determined.

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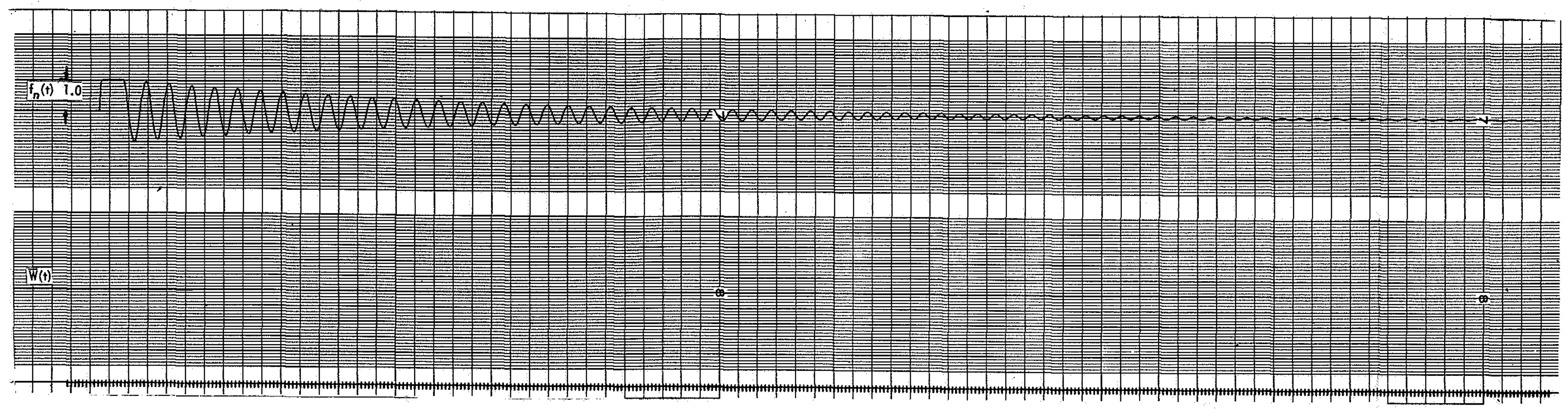

Figure 4-2. Sample Solution of Equation (4.108) without Noise, $\hat{\sigma}^2_{\overline{W}} = 0$.

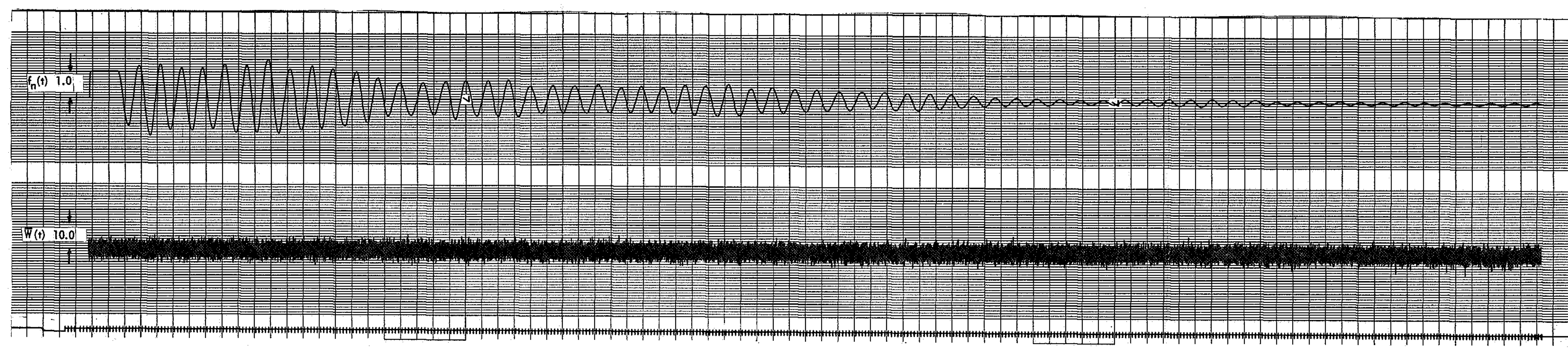

Figure 4-3. Sample Solution of Equation (4.108) with $\hat{\sigma}^2_{\overline{W}} = 2.61$.

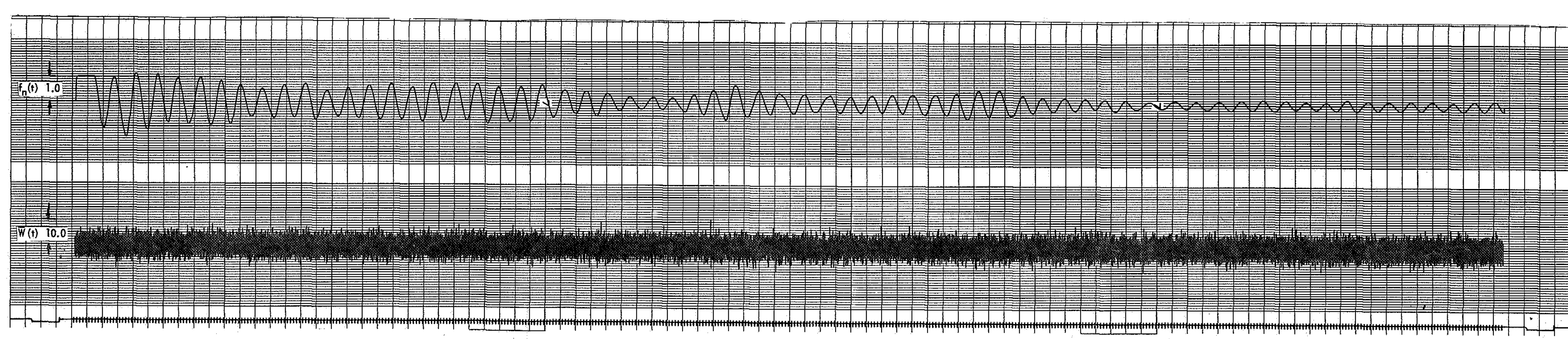

Figure 4-4. Sample Solution of Equation (4.108) with $\hat{\sigma}^2_{\overline{W}} = 5.72$.

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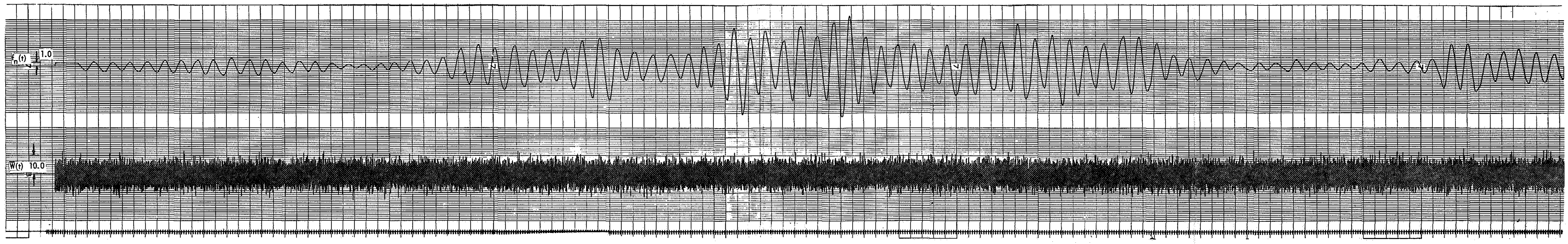

Figure 4-5. Sample Solution of Equation (4.108) with $\hat{\sigma}\frac{2}{W} = 11.44$.

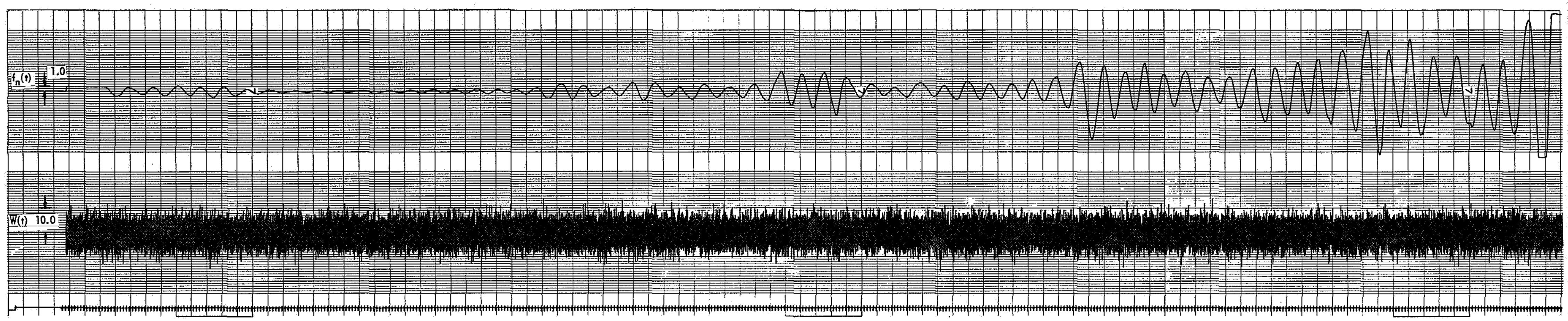

Figure 4-6. Sample Solution of Equation (4.108) with $\hat{\sigma}\frac{2}{W} = 12.01$.

Figure 4-7. Sample Solution of Equation (4.108) with $\hat{\sigma}\frac{2}{W} = 21.27$.

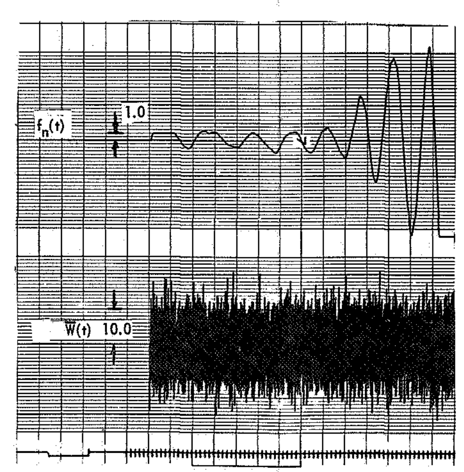

Figure 4-8. Sample Solution of Equation (4.108) with $\hat{\sigma}\frac{2}{W} = 29.98$.

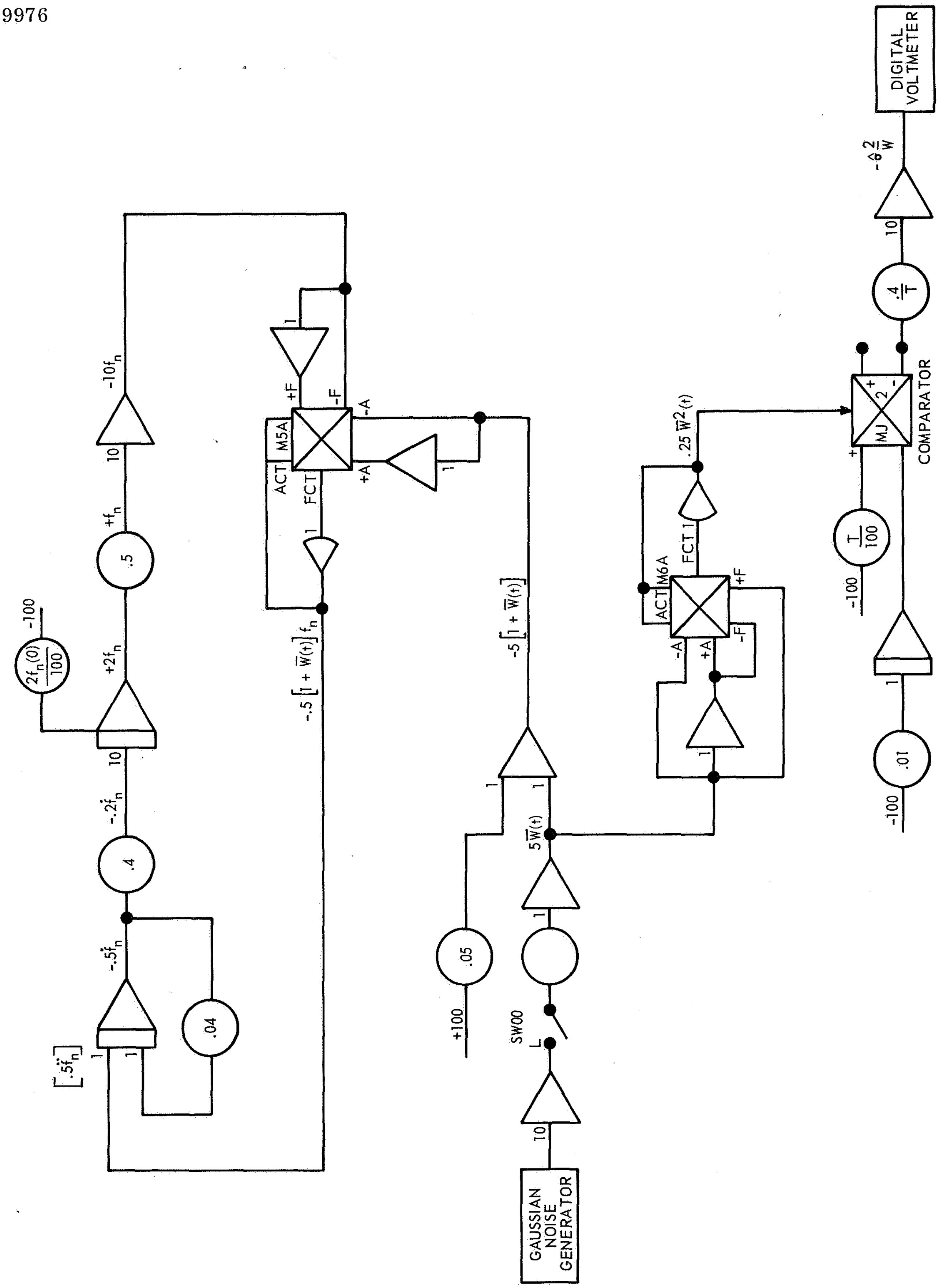

Figure 4-9. Analog Computer Circuit Diagram

It might also be concluded from figures 4-6, 4-7, and 4-8 that the time to saturation (a first passage time) decreases with increasing $\hat{\sigma}^2_W$. This was not true for all sample solutions recorded but did seem to occur on the average.

5. CONCLUDING REMARKS AND RECOMMENDATIONS

Sufficient conditions for asymptotic sample stability of the transverse deflection of a simply supported rod have been obtained for the three types of random longitudinal force considered in sections 4.1.1, 4.1.2, and 4.1.3. An analog computer simulation verified the stability criterion for the case of a Gaussian white noise longitudinal force. These results are given in section 4.2.2.

Since necessary and sufficient stability conditions yield the stability boundary between unstable solutions and stable solutions, a further analysis is needed to determine how conservative are the stability criteria of section 4. Also, the physical interpretation of the stability criteria obtained when using the Gaussian white noise as a parameter in a second order system should be clarified. The results of section 4.1.2 can be extended by determining the actual second moment stability boundaries using Floquet theory.

It would be desirable to extend the method of averaging, discussed in appendix B, to yield a stability criterion which is valid for an infinite interval of time since it could then be applied to nonlinear structural vibration problems.

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APPENDIX A

SAMPLE AND SECOND MOMENT STABILITY OF A FIRST ORDER SYSTEM

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The solution process of a first order differential equation with a noise coefficient can be determined explicitly for both the white noise (the formal derivative of the Brownian motion process) and the wide band physical noise cases. From these solutions, necessary and sufficient conditions for asymptotic sample stability can be determined and a comparison made. Also, the sufficient condition derived from exponential stability of the second moment can be compared to the previous necessary and sufficient conditions.

As was mentioned in section 4.1.1, entirely different stability results may be obtained for white noise or wide band physical noise coefficients. This will be illustrated in the first order example below.

First consider the nonwhite system

$$\dot{x} + [a + f(t)]\, x = 0 \tag{A-1}$$

where a is a constant and f(t) is a wide band mean zero process which satisfies an ergodic property such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\tau)d\tau = E\left\{f(t)\right\} = 0 \tag{A-2}$$

If f(t) is a stationary Gaussian process with an absolutely continuous spectrum, for example, then (A-2) will hold.

The solution to (A-1) is

$$\begin{aligned} x(t) &= x(0)e^{-at-\int_0^t f(\tau)d\tau} \\ &= x(0)e^{-[a + \frac{1}{t}\int_0^t f(\tau)d\tau]\, t} \end{aligned} \tag{A-3}$$

Using (A-2) the exponent of (A-3) becomes in the limit

$$\lim_{t\to\infty} [a + \frac{1}{t}\int_0^t f(\tau)d\tau] = a \tag{A-4}$$

Thus, the limiting solution to (A-1) is

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x(0)e^{-[a + \frac{1}{t}\int_0^t f(\tau)d\tau]t} = \begin{cases} 0 & a > 0 \\ x(0) & a = 0 \\ \infty & a < 0 \end{cases} \tag{A-5}$$

Therefore, a necessary and sufficient condition for asymptotic stability of the sample solutions of (A-1) is

$$a > 0 \tag{A-6}$$

Wong and Zakai [22], however, have shown that in the first order case, a general relationship may be established between the white and non-white cases. Roughly speaking, they proved that if $f^{(i)}(t)$ is a sequence of functions with some given properties which converge to the Gaussian white noise $\dot{B}(t)$ as $i\to\infty$ then the sequence of solutions to

$$\dot{x}^{(i)} + [a + f^{(i)}(t)]x^{(i)} = 0 \tag{A-7}$$

converges in the mean to

$$\dot{x} + [a-\frac{S_W}{2} + \dot{B}(t)]x = 0 \tag{A-8}$$

and not to

$$\dot{x} + [a + \dot{B}(t)]x = 0 \tag{A-9}$$

where S_W is the parameter associated with the variance of B(t), i.e., $E\{B^2(t)\} = S_W t$, and a is a constant. Now consider the Wong-Zakai equation which corresponds to (A-1)

$$\dot{x} + [a - \frac{S_W}{2} + \dot{B}(t)]x = 0 \tag{A-10}$$

It can be shown by using the calculus defined for white noise systems that the solution to (A-10) is

$$x(t) = x(0)e^{-B(t)-at} \tag{A-11}$$

It is known ([23], p. 560) that samples of the Brownian motion process grow like $\sqrt{2t \log_2 t}$ which as $t\to\infty$ is dominated by t. Hence, the asymptotic stability of

(A-10) is determined by the deterministic portion of the exponent of (A-11) and (A-10) will be asymptotically stable if

$$a > 0 \tag{A-12}$$

This is the same stability condition as (A-6).

Now, if one were to proceed from (A-1) in a seemingly natural fashion by replacing the wide band physical noise f(t) by white noise $\dot{B}(t)$, i.e.,

$$\dot{x} + [a + \dot{B}(t)]x = 0 \tag{A-13}$$

then by analogy with (A-10) and (A-11) the solution is

$$x(t) = x(0)e^{-B(t) - (a + \frac{S_W}{2})t} \tag{A-14}$$

which will be asymptotically stable if

$$a + \frac{S_W}{2} > 0 \tag{A-15}$$

In the (a, S_W) plane, the stability of (A-1) and (A-10) can be compared to (A-13).

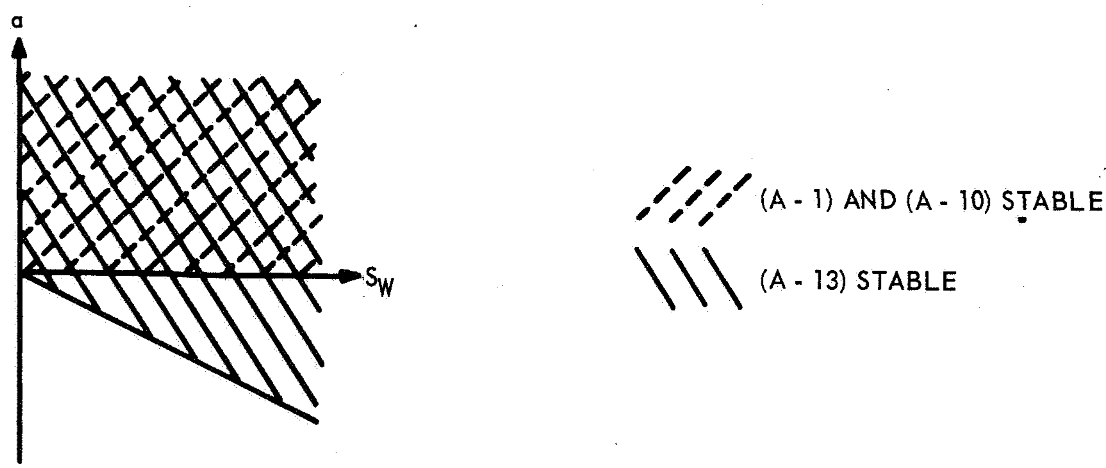

Figure A-1. Sample Stability Boundaries for Equations (A-1), (A-10), and (A-13)

In the first order system, then, the procedure of replacing a wide band noise f(t) by white noise $\dot{B}(t)$ actually increases the stability of the system which certainly counters physical intuition since the variance of $f(t) \to \infty$ as its bandwidth $\to \infty$.

By obtaining a sufficient condition for asymptotic stability from second moment exponential stability and knowing the necessary and sufficient stability conditions, we can get an indication of how conservative the second moment approach is.

Following the procedure of section 4.1.1, we write (A-13) as an equation in increments

$$dx = -axdt - xdB(t) \tag{A-16}$$

for which the derivate moments are

$$\begin{aligned} a(x, t) &= -ax \\ b(x, t) &= S_W x^2 \end{aligned} \tag{A-17}$$

The Fokker Planck equation is

$$\frac{\partial p}{\partial t} = a\frac{\partial (xp)}{\partial x} + \frac{S_W}{2}\frac{\partial^2 (x^2 p)}{\partial x^2} \tag{A-18}$$

and the equation for the n^{th} moment becomes

$$\dot{m}_n = [-na + \frac{S_W}{2} n(n-1)]m_n \tag{A-19}$$

which for $n = 2$ is

$$\dot{m}_2 = [-2a + S_W] m_2. \tag{A-20}$$

$m_2(t)$ is exponentially stable if

$$a - \frac{S_W}{2} > 0. \tag{A-21}$$

In a similar manner the Wong-Zakai equation (A-10) will have a stable second moment if

$$a - S_W > 0 \tag{A-22}$$

In the (a, S_W) plane, (A-21) and (A-22) are

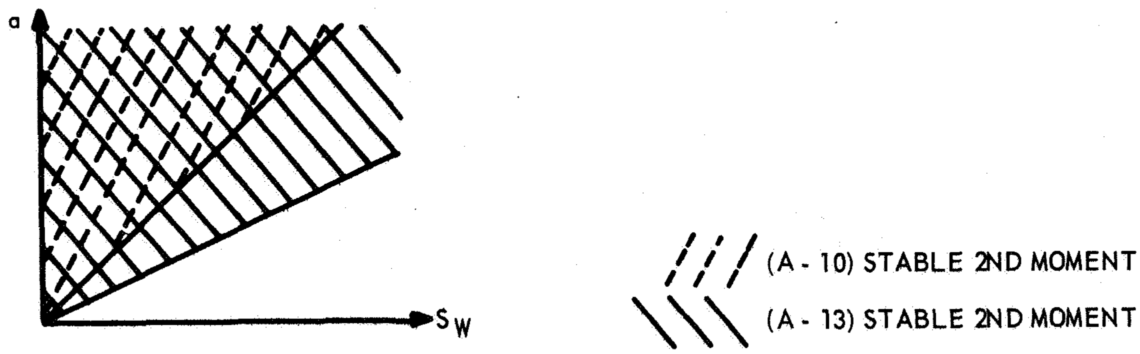

Figure A-2. Sufficient Second Moment Stability Boundaries for Equations (A-10) and A-13).

Compare figures A-1 and A-2. The two stability regions in figure A-1 represent necessary and sufficient conditions for asymptotic sample stability of equations (A-10) and (A-13). The corresponding sufficient conditions for asymptotic sample stability, obtained via the exponential stability of the second moments, are indicated in figure A-2.

APPENDIX B

A BYPRODUCT OF THIS STUDY - THE METHOD OF AVERAGING APPLIED TO AN INVERTED PENDULUM WITH A RANDOMLY OSCILLATING SUPPORT.

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A BYPRODUCT OF THIS STUDY - THE METHOD OF AVERAGING APPLIED TO AN INVERTED PENDULUM WITH A RANDOMLY OSCILLATING SUPPORT.

The interesting fact that a pendulum, whose base is subjected to a periodic displacement, can be stabilized in the inverted position is well known. A natural question to ask is can the pendulum be stabilized with a base motion that is some type of stochastic process?

In [24], conditions for stability were found when the base motion was a finite sum of cosines with random phases and in [17] it was determined that white noise would not stabilize the linearized motion of the pendulum.

The pendulum's equation of motion, (B-6), is nonlinear with the randomly varying base acceleration appearing as a coefficient of the nonlinearity. In the absence of the base acceleration, the system is unstable, i.e., the pendulum falls. Methods that are presently available for determining the stochastic stability of nonlinear systems require the system to be stable in the absence of stochastic coefficients. The approach to be used here, the method of averaging [25], does not require this assumption. This method is quite general and is therefore applicable to many nonlinear dynamic stability problems; thus the analysis presented here is by no means limited to the pendulum example.

The method of averaging approach is as follows. Given a set of nonlinear time varying differential equations, one attempts to reduce these to the "standard" form

$$\dot{x} = \epsilon F(t, x, \epsilon) \tag{B-1}$$

where x and F are n vectors and $\epsilon > 0$ is a small parameter. Then an averaged system of equations is formed

$$\dot{\bar{x}} = \epsilon F_0(\bar{x}) \tag{B-2}$$

where

$$F_0(\bar{x}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T F(t, \bar{x}, \epsilon)\, dt \tag{B-3}$$

Under certain restrictions on F and the existence of the limit in (B-3), the solution of (B-1) and (B-2), starting with the same initial conditions, stay close to each other at each t, in a time interval proportional to ϵ^{-1}, for ϵ sufficiently small.

Thus, if particular solutions to the averaged system (B-2) are stable, then at least for a finite time, the corresponding solutions of (B-1) will be close to these stable solutions.

Stability over an infinite interval of time is the desired property and results of this type have been obtained in the deterministic case when F is assumed to be almost periodic in t. This is not true, in general, for stochastic processes. However, analog simulation of the inverted pendulum with a stochastic base motion strongly suggests stability for an infinite time interval.

Assume that the pendulum may be schematically represented as

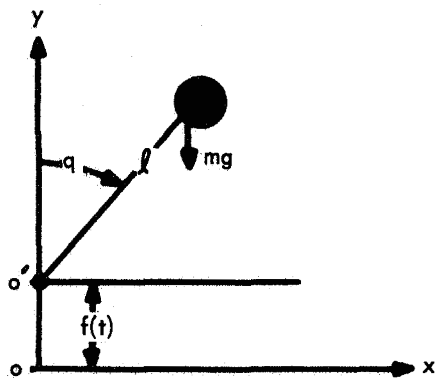

Figure B-1. Schematic Representation of Inverted Pendulum

The kinetic, potential, and dissipation functions may be written as

$$2T = m(\dot{f}^2 - 2\ell \dot{f}\dot{q} \sin q + \ell^2 \dot{q}^2)$$

$$V = -mg\ell(1 - \cos q) + mgf \tag{B-4}$$

$$2D = 2c\ell^2 \dot{q}^2$$

Substituting (B-4) into Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) + \frac{\partial D}{\partial \dot{q}} - \frac{\partial (T - V)}{\partial q} = 0 \tag{B-5}$$

yields

$$\frac{d}{dt}(m\ell^2 \dot{q} - m\ell \dot{f} \sin q) + 2c\ell^2 \dot{q} - mg\ell \sin q + m\ell \dot{f}\dot{q} \cos q = 0$$

or

$$\ddot{q} + \frac{2c}{m}\dot{q} - \left(\frac{\ddot{f}}{\ell} + \frac{g}{\ell}\right)\sin q = 0 \tag{B-6}$$

To put (B-6) in the standard form required by the method of averaging we generate a transformation with the generalized momentum of Hamilton's mechanics and then assume that f is small and rapidly oscillating.

The generalized momentum is defined by

$$p = \frac{\partial L}{\partial \dot{q}} \tag{B-7}$$

where the Lagrangian, L, is

$$L \equiv T - V = \frac{m}{2}\left(\dot{f}^2 - 2\ell\dot{f}\dot{q}\sin q + \ell^2\dot{q}^2\right) + mg\ell(1 - \cos q) - mgf$$

Hence from (B-7)

$$p = m(-\ell\dot{f}\sin q + \ell^2\dot{q}) \tag{B-8}$$

Rearranging (B-8) gives

$$\dot{q} = \frac{p}{m\ell^2} + \frac{\dot{f}}{\ell}\sin q \tag{B-9}$$

Differentiating (B-8) and using (B-6) and (B-9) gives

$$\dot{p} = m(-\ell\ddot{f}\sin q - \ell\dot{f}\dot{q}\cos q + \ell^2\ddot{q})$$

$$= m\left\{-\ell\ddot{f}\sin q - \ell\dot{f}\left(\frac{p}{m\ell^2} + \frac{\dot{f}}{\ell}\sin q\right)\cos q + \ell^2\left[-\frac{2c}{m}\left(\frac{p}{m\ell^2} + \frac{\dot{f}}{\ell}\sin q\right)\right.\right.$$

$$\left.\left. + \left(\frac{\ddot{f}}{\ell} + \frac{g}{\ell}\right)\sin q\right]\right\}$$

$$= -m\left(\frac{p}{m\ell^2} + \frac{\dot{f}}{\ell}\sin q\right)\left(\ell\dot{f}\cos q + \frac{2\ell^2 c}{m}\right) + m\ell g\sin q \tag{B-10}$$

Now assume that f(t) is small and rapidly oscillating in the following sense. Let

$$f(t) = \epsilon w(\epsilon^{-1} t) \tag{B-11}$$

where $\epsilon << 1$ and therefore $\epsilon^{-1} t$ corresponds to a "fast" time. (B-9) and (B-10) become

$$\dot{q} = \frac{p}{m\ell^2} + \frac{\epsilon \dot{w}(\epsilon^{-1} t)}{\ell} \sin q \tag{B-12}$$

$$\dot{p} = -m \left(\frac{p}{m\ell^2} + \frac{\epsilon \dot{w}(\epsilon^{-1} t)}{\ell} \sin q\right) \left(\ell \epsilon \dot{w}(\epsilon^{-1} t) \cos q + \frac{2\ell^2 c}{m}\right) + m\ell g \sin q \tag{B-13}$$

Now transform (B-12) and (B-13) into the "fast" time $\tau = \epsilon^{-1} t$. Using $\frac{d}{dt} = \epsilon^{-1} \frac{d}{d\tau}$ and denoting differentiation with respect to τ by primes gives

$$q' = \epsilon \left(\frac{p}{m\ell^2} + \frac{w'(\tau)}{\ell} \sin q\right) \tag{B-14}$$

$$p' = -\epsilon m \left(\frac{p}{m\ell^2} + \frac{w'(\tau)}{\ell} \sin q\right) \left(\ell w'(\tau) \cos q + \frac{2\ell^2 c}{m}\right) + \epsilon m\ell g \sin q \tag{B-15}$$

However, we note that these last equations are in the standard form for the application of the method of averaging. Hence, if we assume

$$E\{w'(\tau)\} = 0$$
$$E\{(w'(\tau))^2\} = \sigma^2 \tag{B-16}$$

and that $w'(\tau)$ is a sample function from an ergodic process so that ensemble expectations and time averages can be interchanged, then it follows that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T w'(s) ds = 0$$
$$\lim_{T \to \infty} \frac{1}{T} \int_0^T [w'(s)]^2 ds = \sigma^2 \tag{B-17}$$

with probability one.

Averaging (B-14) and (B-15) with respect to explicit functions of time (i.e., $w'(\tau)$) we have using (B-17)

$$\bar{q}' = \epsilon \frac{\bar{p}}{m\ell^2} \tag{B-18}$$

$$\bar{p}' = -\epsilon m \left(\frac{\bar{p}}{m\ell^2} \frac{2\ell^2 c}{m} + \sigma^2 \cos\bar{q} \sin\bar{q} \right) + \epsilon m \ell g \sin\bar{q} \tag{B-19}$$

where $\bar{q}$, $\bar{p}$ are the averaged variables. Converting (B-18) and (B-19) back to the original time t gives

$$\dot{\bar{q}} = \frac{\bar{p}}{m\ell^2} \tag{B-20}$$

$$\dot{\bar{p}} = -m \left(\frac{\bar{p}}{m\ell^2} \frac{2\ell^2 c}{m} + \sigma^2 \cos\bar{q} \sin\bar{q} \right) + m\ell g \sin\bar{q} \tag{B-21}$$

Or in terms of the averaged generalized coordinate $\bar{q}$, we have

$$\ddot{\bar{q}} = \frac{\dot{\bar{p}}}{m\ell^2} = -\frac{1}{\ell^2} \left(\dot{\bar{q}} \frac{2\ell^2 c}{m} + \sigma^2 \cos\bar{q} \sin\bar{q} \right) + \frac{g}{\ell} \sin\bar{q}$$

or

$$\ddot{\bar{q}} + \frac{2c}{m} \dot{\bar{q}} + \left(\frac{\sigma^2}{\ell^2} \cos\bar{q} - \frac{g}{\ell} \right) \sin\bar{q} = 0 \tag{B-22}$$

The averaged equation (B-22) will have stable equilibrium point, $\bar{q} = 0$, $\dot{\bar{q}} = 0$, which corresponds to the pendulum in the vertical position, if

$$\sigma^2 > g\ell \tag{B-23}$$

The method of averaging states that the solutions to the averaged and original equations will be close only for a finite length of time proportional to ϵ^{-1}. Thus it has not been established analytically that the stability conditions (B-11) and (B-23) ensure that the original system (B-6) will be stable for all time. However, the equations (B-12), (B-13) were simulated on an EAI-221 analog computer to see if they could in fact be stable under the conditions (B-11) and (B-23). Roughly speaking these conditions imply that the pendulum base amplitude should be small and have a power spectral density consisting only of "high" frequencies and that the variance of the base velocity should be sufficiently large.

The parameters used in (B-12) and (B-13) were chosen for simplicity as

$$m = 1.0 \quad \text{lb sec}^2/\text{ft}$$

$$\ell = 1.0 \quad \text{ft}$$

$$g = 1.0 \quad \text{ft/sec}^2$$

$$q(o) = 0.0 \quad \text{rad}$$

$$p(o) = 0.01 \quad \text{ft lb sec.}$$

This results in a small amplitude pendulum natural frequency in the down position of 1 rad/sec. The base velocity $\dot{f}(t)$ was obtained from an Elgenco 311A Gaussian noise generator and then filtered in such a manner as to eliminate the frequency content of $f(t)$ in the range 0-25 rad/sec. Approximately fifty sample solutions were obtained for different damping coefficients c and values of $\sigma^2 = E\{[\dot{f}(t)]^2\}$. The six sample solutions shown in figures (B-2) and (B-3) were considered to be typical of those recorded. In all sample solutions recorded for $c = 0.0$, the undamped pendulum fell. Conversely, for $c = 0.1$ all sample solutions were stable.* For the intermediate damping values $c = 0.01$ and $c = 0.02$, over the finite time intervals during which the sample solutions were recorded, there were stable and unstable sample solutions.

Some conclusions can be made from the analog computer results. By comparing figures (B-2) and (B-3) we see that increasing the variance of the noise σ^2 tends to increase the average frequency of the pendulum oscillation. The "stability" criterion, (B-11) and (B-23), obtained from the method of averaging, is independent of the damping coefficient c; it is obvious, however, from figures (B-2) and (B-3) that the stability is dependent on damping.

To illustrate the fact that the averaged equations remain close to the original equations for a finite interval of time, the averaged equations (B-20) and (B-21) were simulated on the analog computer simultaneously with (B-12) and (B-13). The parameters used in (B-12), (B-13), (B-20), and (B-21) were identical to those above with $c = 0.01$ and $\sigma^2 = 4.18$. The results are shown in figure (B-4).

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*Obviously, infinite time stability of stochastic systems can not be proved via analog simulation. However, the form of the sample solution corresponding to $c = 0.1$ shown in Figure (B-2) strongly suggests infinite time stability.

FOLDOUT FRAME 1

FOLDOUT FRAME 2

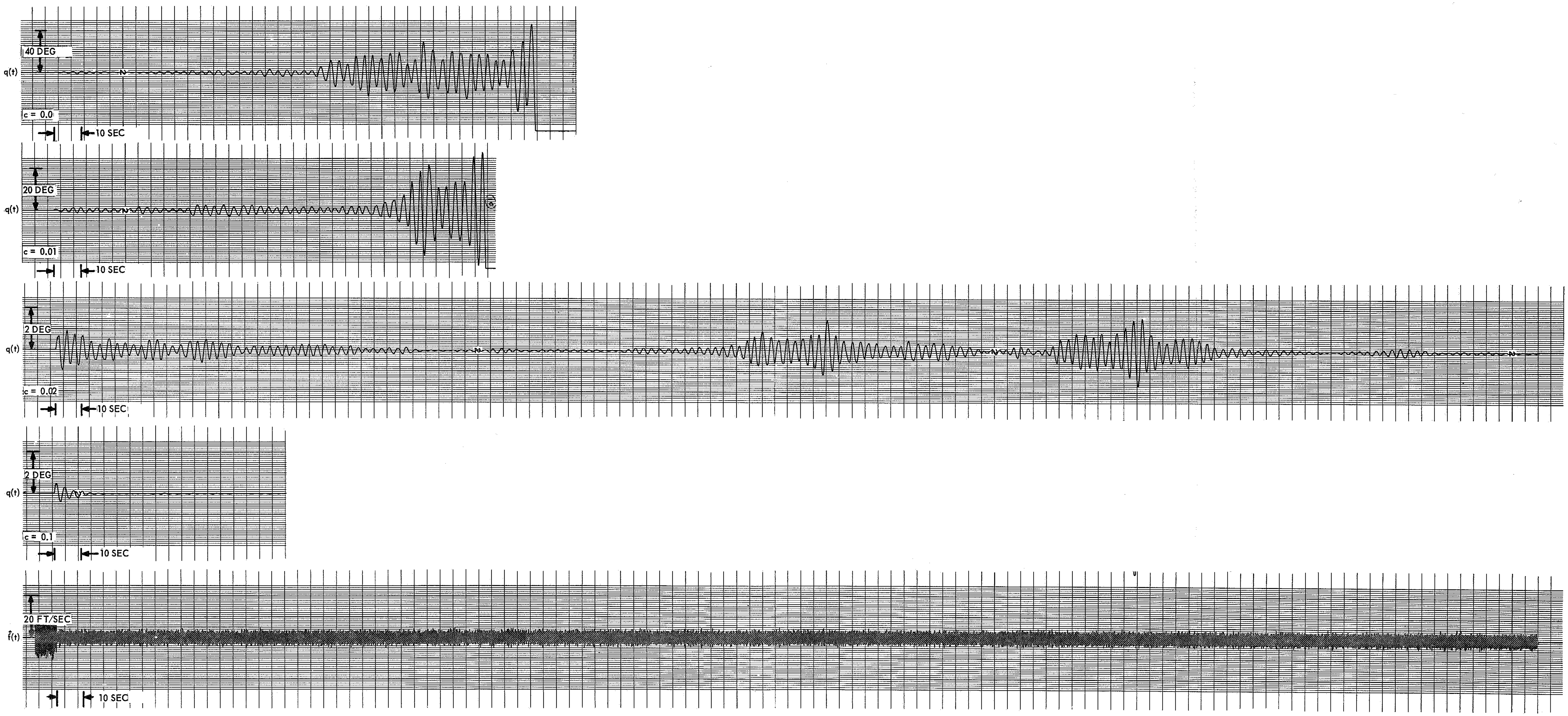

Figure B-2. Sample Solutions of (B-12) and (B-13) for $\sigma^2 = 1.90$ and Various Damping Coefficients c and a Typical Noise Sample $\dot{f}(t)$.

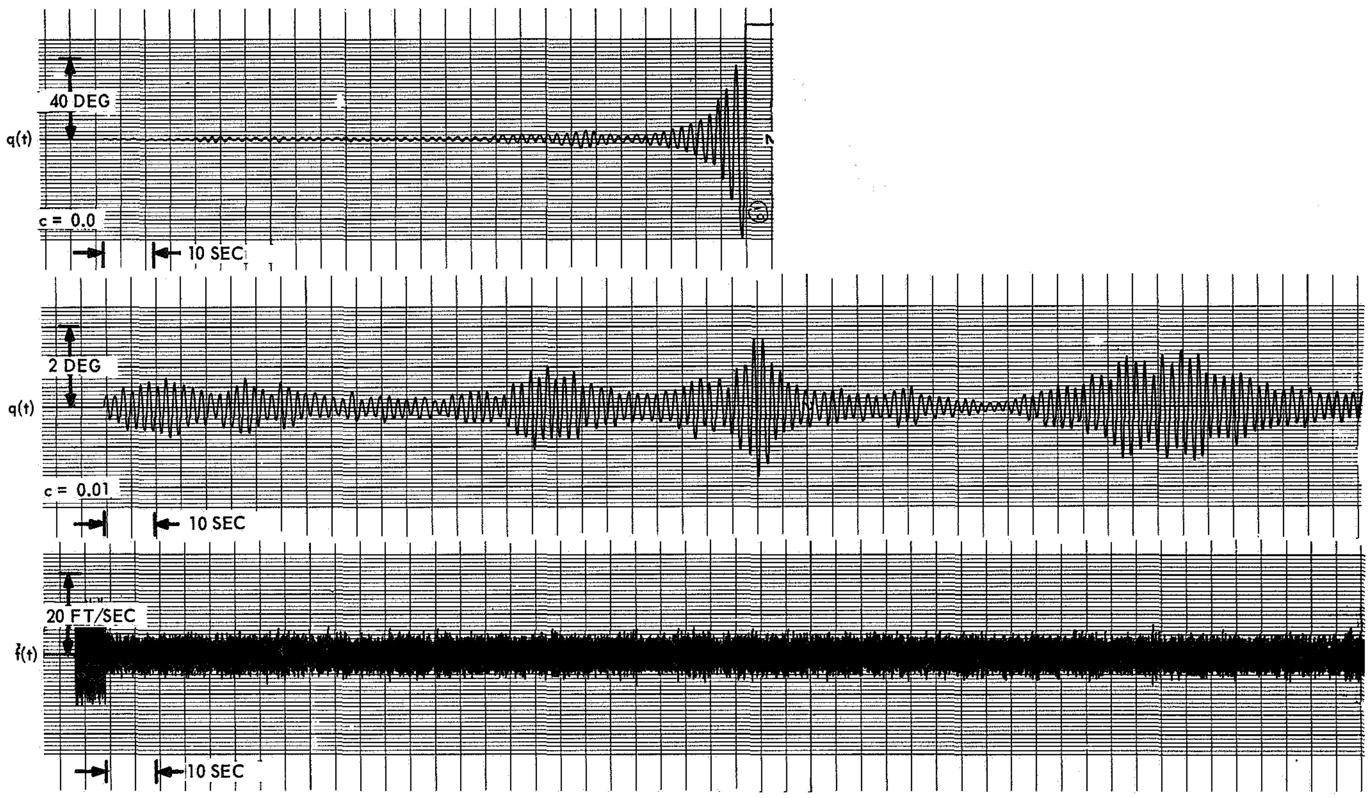

Figure B-3. Sample Solutions of (B-12) and (B-13) for $\sigma^2 = 4.18$ and Various Damping Coefficients c and a Typical Noise Sample $\dot{f}(t)$.

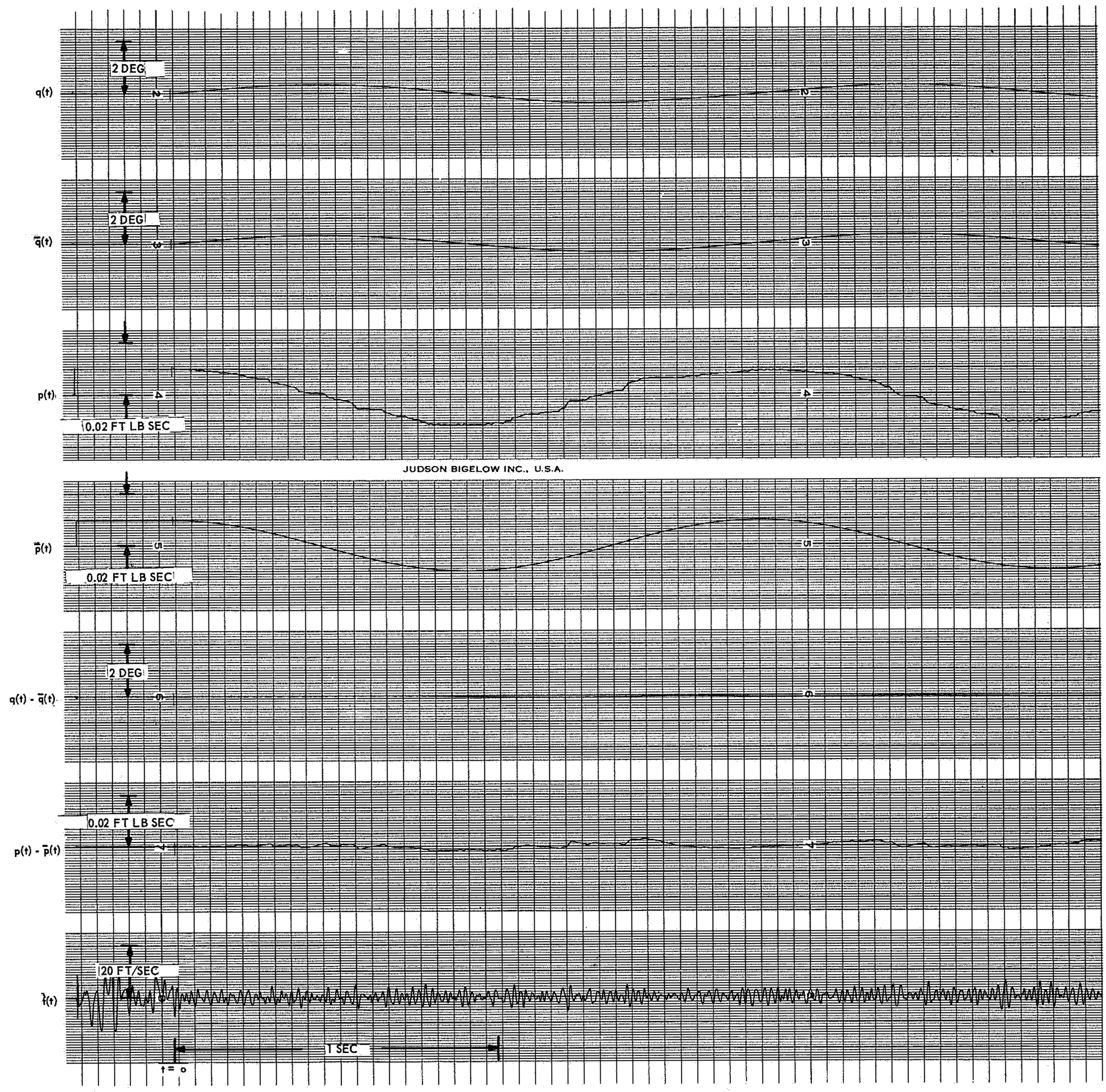

Figure B-4. Sample Solution of Equations (B-12) and (B-13) and the Averaged Equations (B-20) and (B-21) for $\sigma^2 = 4.18$ and $c = 0.01$.